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# CONTROL AND SIMULATION OF HYBRID SYSTEMS *

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***Abstract*** We consider simulation of hybrid systems consisting of continuous time smooth systems and relays. We discuss how a simulation program should detect the structural possibility of infinite fast mode switches (sliding) and help the user introduce new so called induced modes. We also analyze the problem of how to define the dynamics on the induced modes. Finally we study the different possible stable sliding motion around an intersection of two transversal switching surfaces.

***Keywords*** Variable Structure Systems, Sliding Mode Behavior, Cyclic Control, Hysteresis, Hybrid Systems, Mode Switched Systems, Simulation Tools, Modeling, Two-relay Systems.

## 1. INTRODUCTION

Today most investigations of hybrid systems are done by simulation. Even though much research effort is put into the the field of hybrid systems this situation is not likely to change in the near future. It is therefore crucial to have good simulation tools for hybrid systems. Today, there are several simulation packages that allow for the mixture of continuous variables and discrete variables but the simulation performance is often poor. One important problem is simulation of models where so called sliding mode behavior might arise, i.e. there might be infinite many mode switches in a finite time interval.

A typical hybrid control system is seen in Fig 1. Mode switches in this system can be caused either by switching between different controllers or different operation modes of the plant.

There are many models of hybrid control systems. A common feature is that the state space $S$ has both discrete and continuous variables, for example $S \subset R^n \times Z^m$. The models proposed by different authors differ in definition of and restrictions on dynamic behavior. There is not yet any agreement on what constitutes the most fruitful compromise between model generality and powerfulness. For a review of different approaches see e.g. [2] or [9]. Our class of systems is a special case of the Differential Automata described in [13]:

$$\begin{aligned} \dot{x} &= f(x(t), q(t)), && x \in \mathcal{R}^n \\ q(t) &= \nu(x(t), q(t^-)), && q \in \mathcal{Z}^{m+} \end{aligned} \tag{1}$$

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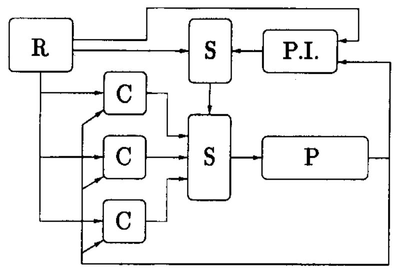


**Fig. 1** A hybrid control system, R-reference generator, C-controller, S-scheduler, S-switch, P-plant, P.I.-performance index

where $x$ denotes the continuous and $q$ the discrete variables. The model does not allow for autonomous or controlled jumps. Hybrid models like these are often represented with a graph, see Fig 2. Here each of the nodes represent a mode of the system. Associated with each mode is a dynamic equation and jump conditions. In this paper we will let the state of a set of relays govern the modes. We will

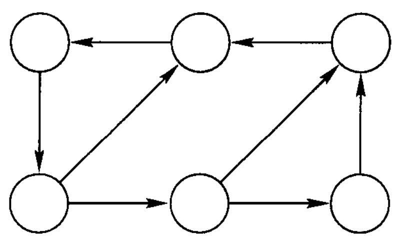

**Fig. 2** Typical graph of hybrid system

specialize (1) to the following system where the modes are determined by the state of $m$ relays:

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x) \\ u_i &= \text{sgn}(y_i), \quad i = 1, \dots, m \end{aligned}$$

where $x \in R^n$, $u, y \in R^m$ and $f, h \in C^1$. Here sgn(x) equals 1 if $x > 0$, equals $-1$ if $x < 0$ and is undefined if $x = 0$. The system dynamics is hence undefined on the surface $h(x) = 0$.

For many of the proposed hybrid models restrictions are introduced to prevent infinitely fast switches between the discrete modes. For instance in [13] the distance between any two sets with different discrete transitions is bounded away from zero and the next set from which another discrete transition takes place is at least a fixed distance away. After a discrete transition one is in an open set on which the dynamics are well defined. In practical situations it is however common to have so called sliding modes, i.e. cycles of infinitely fast changes of modes. Some design methods may lead to this behavior, see [6]. Simulating such systems is hard. In this paper we will discuss automatic detection of the structural possibility of fast switching and discuss how new "induced mode" can be introduced. We will then treat the two-relay case in in $R^3$ in more detail.

As an example consider the system

$$\begin{aligned} \dot{x}_1 &= \cos(u) \\ \dot{x}_2 &= -\sin(u) \\ y &= x_2 \\ u &= \operatorname{sgn}(y) \end{aligned}$$

which is a simple model of a vehicle with control system to drive along the $x_2 = 0$ axis. The control system controls the steering direction $u$ of the car with a relay. The system can be aggregated into the following hybrid automaton, described in Fig. 3. Note that the transitions are not forced and that

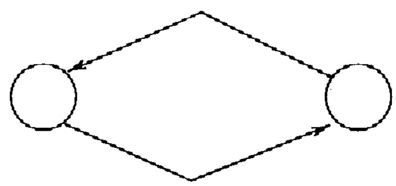


**Fig. 3** A system with two modes where fast switching occurs

fast switching will occur on the surface $x_2 = 0$ where the dynamics is not well defined. It would be advantageous if a simulation program might, automatically, detect the structural possibility of fast switching in large systems with several relays, extend the system with induced modes so that the new system has only forced transitions without fast switching and describe how these modes could inherit dynamics from the basic system. For the car-example above the new automaton should have one induced mode and only forced transitions, see Fig. 4. There are several possible definitions of the inherited dynamics. To determine which is physically motivated, more modeling is typically needed.

### 1.1 *Inheritance of dynamics*

Sliding motion along a surface, such as $x_2 = 0$ in the car example was studied by e.g. Filippov, see [5], [14] or [3]. For a smooth surface described by the intersection of $m$ smooth surfaces $y_i(x) = 0,\ i = 1, \ldots m$ the dynamics along the intersection can be defined in at least two possible ways.

DEFINITION 1—FILIPPOV CONVEX DEFINITION

$$\dot{x} \in \operatorname{co}\{f(x,u) \ : \ u_i = -1 \text{ or } 1,\ i = 1, \ldots, m\} \tag{2}$$

$$\text{such that} \qquad \frac{d}{dt} y_i(x) = \frac{\partial y_i}{\partial x}\dot{x} = 0, \quad i = 1, \ldots, m \tag{3}$$

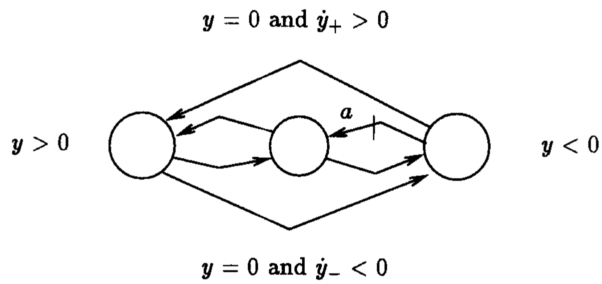


**Fig. 4** System with one induced mode. Only one induced transition is indicated, $a : (y = 0 \text{ and } \dot{y}_+ < 0)$

The differential inclusion (2) can be written as

$$\dot{x} = \sum_{u \in \{-1,1\}^m} \alpha_u(t) f(x,u)$$

where $\sum_i \alpha_u(t) = 1$ and $\alpha_u(t) \geq 0$. This definition has roots in optimal control under the name of extended control and from generalized derivatives in non-smooth analysis, see [3]. The definition can be motivated by a limiting process where the relay equations are replaced by $u_i(t) = \text{sgn}(y_i(t - \epsilon_i))$, where $\epsilon_i \to 0+$. This can for example be a good approximation if the relays are implemented on a digital computer.

An alternative definition of the sliding mode dynamics is the following.

Definition 2—Filippov equivalent control
Relax the $m$ discrete variables in $u$ to continuous variables in $[-1,1]^m$ and find $u_{eq}$ such that

$$\dot{x} = f(x, u_{eq}) \tag{4}$$

$$\text{such that} \quad \frac{d}{dt} y_i(x) = \frac{\partial y_i}{\partial x} \dot{x} = 0, \quad i = 1, \ldots, m \tag{5}$$

This definition can be motivated by a limiting process where each relay is approximated by a continuous function $\text{sgn}_\epsilon(x)$. This can be a good approximation if the relays are implemented with analog components. The two definitions coincide if $f(x,u)$ is affine in $u$.

For the example with the car, the convex definition gives the motion $\dot{x}_1 = \cos(1)$ along the switching line $x_2 = 0$. The definition of equivalent control gives $\dot{x}_1 = \cos(0) = 1$. Both definitions can be natural candidates for the physical behavior of the car, depending e.g. on if the control system is implemented on a digital or analog computer.

If there are more than one relay, i.e. if $m > 1$ then equations (2) and (3) are not sufficient to define the sliding motion uniquely. The case with several relays is in fact not very well understood. The "physical" sliding motion will depend on the salient features of the different relays, e.g. which relay is the fastest. A simple case with two relays is investigated in [11]. This analysis is extended in Section 3. Another case of non uniqueness is given by non-transversal sliding, i.e. if $\frac{\partial y_i}{\partial x} f(x,u) = 0$ forall $u$.

1.2 *Nontranversal sliding*

In the literature it is common to assume transversal intersection of vector fields and switching surfaces. Nontransversal intersections can however arise quite naturally and should not be considered degenerate or nongeneric. To see this we extend the car example with a third equation describing sensor dynamics

$$\begin{aligned}
\dot{x}_1 &= \cos u \\
\dot{x}_2 &= -\sin u \\
\dot{x}_3 &= -x_3 + x_2 \\
y &= x_3 \\
u &= \mathrm{sgn}(y).
\end{aligned}$$

The switching surface is given by $x_3 = 0$. There will be nontransversal sliding on the line $x_2 = x_3 = 0$. Note that $\frac{\partial y}{\partial x} \cdot f(x,u) = 0$ on this line for all $u$. To define the sliding dynamics it can be motivated to extend the previous definitions in the following way: Differentiate $y(x)$ with respect to $t$ until it is possible to solve for $u$. For the car example the convex definition is given by equations (2), $\dot{y} = 0$ and $\ddot{y} = 0$. The equivalent control is given by (4), $\dot{y} = 0$ and $\ddot{y} = 0$. (This gives the same sliding behavior in the $x_1$ direction as without sensor dynamics.)

Necessary and sufficient conditions for existence of higher order switching for systems with one relay is to our knowledge unknown. A necessary condition is that $\frac{\partial}{\partial u} y^{(k)} < 0$ where $k$ is the smallest integer such that $\frac{\partial}{\partial u} y^{(k)} \neq 0$. If $f(x,u) = a(x) + b(x)u$ the condition can be written $L_b L_a^{k-1} h(x) < 0$. Here $L_b$ denotes the Lie-derivative of in the direction of $b(x)$. Hence $k$ is the nonlinear relative degree from the output from the relay to the input. If the relative degree $k$ is greater than 1 the intersection is nontransversal. Transversal intersections are hence generic only in the same weak sense as "linear systems are generically of relative degree 1".

1.3 *Stable nontransversal sliding of degree k*

Sufficient conditions for nontransversal sliding are hard, since stability of the switching line must also be studied. If we for instance change the sensor dynamics equation above to $\dot{x}_3 = x_3 + x_2$ the switching line $x_2 = x_3 = 0$ will be unstable and there will not be any sliding motion along the line. We have been able to solve the case with $k = 2$. This will be presented in a later paper.

## 2. STRUCTURAL DETECTION OF CYCLES – INDUCED MODES

It is of great help if the simulation tool is able to check for possible simulation problems before the actual simulation is started. Specifically, the simulation tool should warn if infinitely fast switching between modes are possible. The tool should then determine which variables are involved in such fast cycles and suggest new induced modes. The construction of an appropriate model for simulation will typically be an iterative process with interaction between the user and the simulation tool. In a simulation environment such as Omsim, see [1], it is possible to do structural analysis of the equations before simulation. Such a structural analysis can be achieved by efficient graph methods. Distinction is then made only between zero and nonzero coefficients. An example of such structural analysis is block-lower triangular (BLT) partion of the problem with respect to the variables. This has been used to detect algebraic loops of minimal dimension, see [12] and [4]. Another example is Pantelides' algorithm, see [10] and [7] to determine suitable forms of integration of high-index DAE problems. A BLT partitioning of the car example above would reveal that $u$ structurally influences $x_2$, and therefore also indirectly $x_3$ i.e. the input to the relay. Hence there might exist a fast sliding mode involving $u, x_2$ and $x_3$. This can be nontrivial to see if the loop is part of a larger system.

Since structural analysis only gives necessary conditions for fast switching, we can not guarantee that switching actually will occur during simulation. We now describe the structural analysis to determine if fast switching *may* occur for problems with multiple relays. We do this by considering a simple example, from which the general algorithm should be obvious.

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, u_1) \\
y_2 &= h_2(x_1) \\
\dot{x}_2 &= f_2(u_2) \\
y_1 &= h_1(x_2) \\
u_1 &= \mathrm{sgn}(y_1) \\
u_2 &= \mathrm{sgn}(y_2)
\end{aligned}$$

The actual form of the functions $f_i$ and $h_i$ does not matter in this discussion since only structural dependencies are analyzed. After adjoining the equations $\dot{x}_i = \frac{d}{dt}x_i$ the corresponding structure Jacobian is given by

| | $\dot{x}_1$ | $x_1$ | $\dot{x}_2$ | $x_2$ | $u_1$ | $y_1$ | $u_2$ | $y_2$ |
|---|---|---|---|---|---|---|---|---|
| $f_1$ | ⋆ | ⋆ | | | ⋆ | | | |
| $x_1$ | ⋆ | ⋆ | | | | | | |
| $h_2$ | | ⋆ | | | | | | ⋆ |
| $f_2$ | | | ⋆ | | | | ⋆ | |
| $x_2$ | | | ⋆ | ⋆ | | | | |
| $h_1$ | | | | ⋆ | | ⋆ | | |
| $\mathrm{sgn}_1$ | | | | | ⋆ | ⋆ | | |
| $\mathrm{sgn}_2$ | | | | | | | ⋆ | ⋆ |

In this case there is a loop of the form $u_1, \dot{x}_1, x_1, y_2, u_2, \dot{x}_2, x_2, y_1, u_1$ which is easy to find using a straight forward graph algorithm. The induced mode should determine the dynamics on the set $y_1(x) = y_2(x) = 0$. The Filippov equivalent control dynamics can be determined by substituting the relay equations with the equations $y_1(x) = y_2(x) = \dot{y}_1(x) = \dot{y}_2(x) = 0$ and solving for $u$, (if this is possible, otherwise higher derivatives of $y_1(x)$ and/or $y_2(x)$ must be computed). The Filippov convex control is in general not uniquely defined by the new equations. It will depend on the salient features of the relays.

### 2.1 *Manual determination of induced modes*

Sometimes an extended model can be suggested automatically but in most cases we need to manually add some modes to the hybrid system. We call these new modes ***induced modes.*** In the "convex definition" of sliding, the dynamics in these induced modes are linear combinations of the vector fields adjacent to the surfaces. This means that the solution can be written as

$$\dot{x} = \sum_{u \in \{-1,1\}^m} \alpha_u f(x, u) \tag{6}$$

To be able to calculate the $\alpha_i$'s it is not sufficient to use a perfect relay model for the switch. We will look at some additional equations to determine the $\alpha_i$'s for special cases further down.

***Example*** To illustrate the mode inducement procedure we will study a problem with two relays and four vector-fields. The four constant vector-fields are as in Fig. 6, right hand side. On the plane, $x_2 = 0, x_1 > 0$, there is a sliding motion. On the line, $x_2 = 0, x_1 = 0$ the motion is not well defined. Fig. 5 shows the analysis procedure step by step. In the start we have four different modes, 1–4, with four different sets of dynamics. Analysis of the jump conditions will reveal that there is a possibility for a sliding motion. Both transitions $T_{14}$ and $T_{41}$ can be enabled simultaneously. Therefore we generate a new mode, 5. In this mode the dynamics can be determined by calculating, for example, the Filippov convex combination solution. It is possible to go from modes 1 and 4 to mode 5 but not in the other direction. Furthermore, the exit transition from mode 1 to mode 2 will be inherited by the new mode. After the new induced mode has been introduced we do the transition analysis again and find that there is a loop of enabled transitions. This will lead to fast cycling. Again we generate a new mode. In general the dynamics will be as in Eq. 6. In this example the dynamics is not a unique combination of the vector-fields $f_1$–$f_4$. To resolve this problem we need more modeling information. This is now illustrated in the case of systems with two relays.

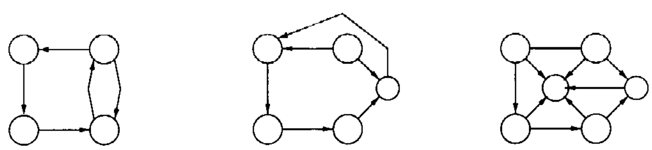

**Fig. 5** Film of mode inducement

## 3. TWO-RELAY SYSTEMS

In this section we will study a simplified version of Eq. 1 with two relays and constant vector-fields. The dynamics are again

$$\dot{x} = \sum_{u\in\{-1,1\}^2} \alpha_u f(x,u)$$

where $x \in R^3$. After a state transformation we will classify all the stable motions in the $x_1$-$x_2$-plane. The inputs of the relays are then such that switching occur on the lines,

$$\begin{aligned}
S_1(x) &: x_1 = 0 \\
S_2(x) &: x_2 = 0 \\
S_1^+(x) &: x_1 = 0, \quad x_2 > 0 \\
S_1^-(x) &: x_1 = 0, \quad x_2 < 0 \\
S_2^+(x) &: x_2 = 0, \quad x_1 > 0 \\
S_2^-(x) &: x_2 = 0, \quad x_1 < 0
\end{aligned}$$

This means that $u_i(x) = sgn(x_i)$. With two relays it is possible to have six different stable motions in the $x_1$-$x_2$-plane. For typical vector-field patterns see Fig 6 to Fig 8. Stability is here defined as

$$x(t) \rightarrow \mathcal{S}_{12} \text{ as } t \rightarrow \infty \quad \text{where} \quad \mathcal{S}_{12} = \mathcal{S}_1 \cap \mathcal{S}_2.$$

All of these stable cases may cause difficulties to a simulation tool. As an example, simulating a systems of type *rrrs*, see Fig 6 right, is difficult on $\mathcal{S}_2^+$ and especially on $\mathcal{S}_{12}$. The dynamics on $\mathcal{S}_2 \setminus \mathcal{S}_{12}$ can be calculated using Filippov solutions. In Fig 6 to Fig 8 right, we have indicated where new modes may be needed. The algorithm to generate and use these figures is the following.

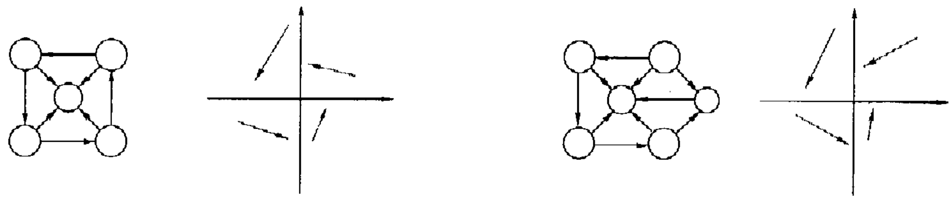

**Fig. 6** Type rrrr and type rrrs

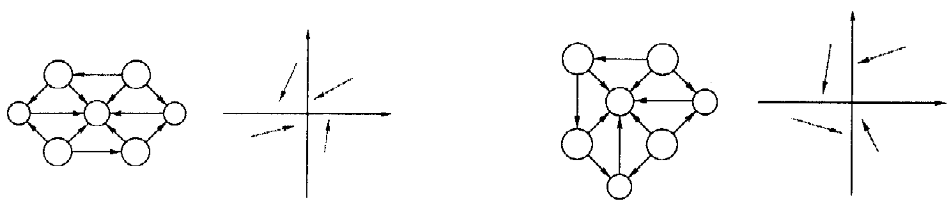

**Fig. 7** Type rsrs and type rrss

ALGORITHM 1—MODE INDUCEMENT ALGORITHM

1 Calculate the switching manifolds associated with each relay, $\mathcal{S}_1^{\pm}$ and $\mathcal{S}_2^{\pm}$. The original modes forme a hypercube

2 Determine manifold intersections.

3 Generate induced modes for each intersection. For the two-relay case five induced modes are generated.

4 Remove modes that are not attractive, i.e. give vector-fields that point towards the manifold associated with the mode.

Point 4 above can be done at different occasions depending on the problem.

1 Constant vector-fields – before actual simulation, right after structure check.

2 Variable vector-fields – during simulation.

### 3.1 *Classification tools*

To facilitate the work on point 4 in Algorithm 1 we will work with some matrices describing the problem as in [11]. Introduce the matrix $V$ containing the four vector-fields

$$V = \begin{bmatrix} f_{11} & f_{21} & f_{31} & f_{41} \\ f_{12} & f_{23} & f_{32} & f_{42} \\ f_{13} & f_{23} & f_{33} & f_{43} \end{bmatrix} \tag{7}$$

and a coefficient matrix $C$ that will be used for checking the sliding equations

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ f_{11} & f_{21} & f_{31} & f_{41} \\ f_{12} & f_{23} & f_{32} & f_{42} \end{bmatrix}. \tag{8}$$

Will will also need the projection matrices

$$P_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \tag{9}$$

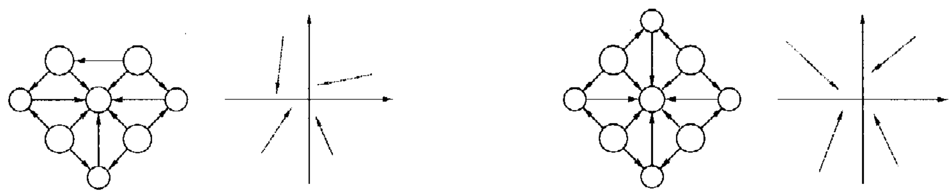

**Fig. 8** Type rsss and type ssss

### 3.2 *Sliding modes*

In the case when the sliding motion involves only two modes the new dynamics can be generated fairly easy. As additional modeling information all that is needed is what kind of Filippov solution to use.

### 3.3 *Fast cycling or multi sliding*

If a new mode is induced from several, i.e. more than two, other modes then we need more information of the salient features of the relay approximation in order to generate the new dynamics. In our example with two relays there is a difficulty on $\mathcal{S}_{12}$. Here there is an ambiguity in the sliding velocity. Four vector-fields are to be balanced by only three equations. With $\alpha = [\,\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4\,]^T$. The necessary conditions to stay on $\mathcal{S}_{12}$ is

$$P_{12}V\alpha = [\,0 \quad 0\,]^T. \tag{10}$$

Solutions to Eq. 10 can be written as $\alpha^0 + p\alpha^N$, where $\alpha^N$ satisfies $C\alpha^N = 0$ and where $\alpha^0 + p\alpha^N \geq 0$. A first step in determining the velocity on the intersection is to calculate the minimum and the maximum velocities. The can be done by solving a linear programming problem of the form

$$\max_p \quad P_3V(\alpha^0 + p\alpha^N) \tag{11}$$

For special cases of $V$ the minimum and the maximum coincide. This happens when $f(x,u)$ can be written as $f(x,u) = f_0(x) + [\,f_I(x) \quad f_{II}(x)\,]\,[\,u_1 \quad u_2\,]^T$, i.e. when $f(x,u)$ is affine in $u$.

### 3.4 *Relays with hysteresis*

One possibility to get a unique solution is to use relays with hysteresis. We will use this method to define a unique solution to the Eq. 6 for some of the six cases.

***Case ssss*** The case, *ssss*, with sliding motion on all four half-surfaces was treated in [11], In this paper two relays with different levels of space hysteresis were used. This means that the input to the relay has to exceed a certain level, $\epsilon$, before it switches. It is straight forward to extend this to any combination of time and space hysteresis. Time hysteresis means that after the input passes zero the relay switches after a certain time, $\tau$. If a fast relay, ($\tau_1$ or $\epsilon_1$ small) is used over the $x_1$-axis and a slower relay, ($\tau_2$ or $\epsilon_2$ larger) is used over the $x_2$-axis then upon letting $\tau_1;\epsilon_1 \to 0$, $\tau_2;\epsilon_2 \to 0$ and $\frac{\tau_1;\epsilon_1}{\tau_2;\epsilon_2} \to 0$ the multi sliding velocity is given uniquely by

$$\dot{x} = F^{1423} + \mathcal{O}(\frac{\tau_1,\epsilon_1}{\tau_2,\epsilon_2}), \tag{12}$$

where $F^{1423}$ is the unique result of taking Filippov convex combinations

$$\begin{aligned} F^{14} &= \lambda_1 f_1 + \lambda_4 f_4, \quad P_1 F^{14} = 0 \\ F^{23} &= \lambda_2 f_2 + \lambda_3 f_3, \quad P_1 F^{23} = 0 \\ F^{1423} &= \lambda_{14} F^{14} + \lambda_{23} F^{23}, \quad P_2 F^{23} = 0 \\ 1 &= \lambda_1 + \lambda_4 = \lambda_2 + \lambda_3 = \lambda_{14} + \lambda_{23} \end{aligned} \tag{13}$$

(14)

and $\mathcal{O}(\frac{\tau_1, \epsilon_1}{\tau_2, \epsilon_2})$ means any of the combinations, $\mathcal{O}(\frac{\tau_1}{\tau_2}), \mathcal{O}(\frac{\tau_1}{\epsilon_2}), \mathcal{O}(\frac{\epsilon_1}{\tau_2}), \mathcal{O}(\frac{\epsilon_1}{\epsilon_2})$. Making the other relay faster leads to a corresponding sliding motion of

$$\dot{x} = F^{1234} + \mathcal{O}(\frac{\tau_2, \epsilon_2}{\tau_1, \epsilon_1}) \tag{15}$$

Given the information of the relative speed of the relays it is hence possible to uniquely generate the dynamics to use in the induced mode.

### 3.5 *Case rrrr*

This is the pure cyclic control case. Outside the set $\mathcal{S} = P_{12}x = 0$ the controllers are used in the fixed sequence, $c_1$–$c_2$–$c_3$–$c_4$. The solution is well defined and depends only on the initial condition. Stability is checked by operations on the matrix $V$. The system is stable if

$$c = \frac{f_{41}}{f_{42}} \frac{f_{32}}{f_{31}} \frac{f_{21}}{f_{22}} \frac{f_{12}}{f_{11}} < 1. \tag{16}$$

Switching will be faster and faster as the state approach the the set defined in Eq. 10. Finally, after a finite time $t^\dagger$, the trajectory will reach the set $\mathcal{S}$. The time to reach the set $\mathcal{S}$ is given by

$$t^\dagger = \lim_{k \to \infty} t_k = \sum_{j=1}^{\infty} c^{j-1} t_1 = (1-c)^{-1}. \tag{17}$$

After $t = t^\dagger$ the solution is no longer unique and most standard simulation tools fail.

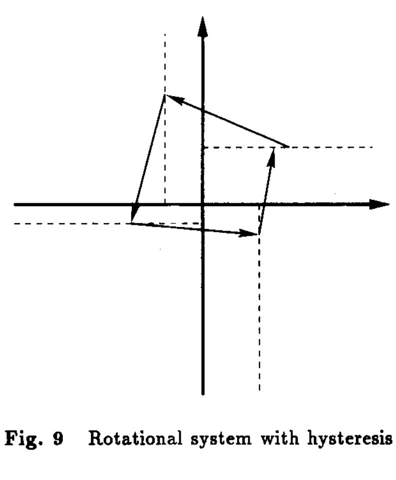

**Fig. 9** Rotational system with hysteresis

Again the dynamics depends on the salient features of the relays. If we introduce hysteresis, see Fig. 9, the solution will tend to a unique limit cycle. Using the switching points of this limit cycle we can generate the dynamics to use in the new induced mode. A natural definition of the dynamics on $S$ would be obtained by letting $\epsilon_i$ tend to zero. But as in the case *ssss* the limit $\lim_{\epsilon\to 0} P_3 V\alpha(\epsilon)$ in general depends on the relative size of the $\epsilon_i$'s.

## 4. SUMMARY

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