Control of Nonholonomic Systems

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1 Introduction

When the generalized velocity of a mechanical system satisfies an equality condition that cannot be written as an equivalent condition on the generalized position, the system is called a nonholonomic system [1, 2]. Nonholonomic condition may arise from constraints such as pure rolling of a wheel or from physical conservation laws such as the conservation of angular momentum of a free floating body.

Nonholonomic systems pose a particular challenge from the control point of view, as any one who has tried to parallel park a car in a tight space can attest. The basic problem involves finding a path that connects an initial configuration to the final configuration and satisfies all the holonomic and nonholonomic conditions for the system. Both open loop and closed loop solutions are of interest: open loop solution is useful for off-line path generation, closed loop solution is needed for the real time control.

Nonholonomic systems typically arise in the following classes of systems:

1. *No-slip constraint.*

   Consider a single wheel rolling on a flat plane (see Figure 1). The no slippage (or pure rolling) contact condition means that the linear velocity at the contact point is zero. Let $\vec{\omega}$ and $\vec{v}$, respectively, denote the angular and linear velocity of the body frame attached to the center of the wheel. Then the no slippage condition at the contact point can be written as

$$\vec{v} - \ell\vec{\omega} \times \vec{z} = 0. \quad (1)$$

   We will see later that part of this constraint is non-integrable (i.e., not reducible to a position constraint) and, therefore, nonholonomic.

   In modeling the grasping of an object by a robot hand, the so–called soft finger contact is sometime used. In this model, the finger is not allowed to rotate about the local normal, $\vec{z} \cdot \vec{\omega} = 0$, but free to rotate about the local x and y axes. This velocity constraint is non-integrable.

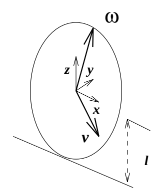

Figure 1: A Wheel with No-Slip Contact

The dynamic equations of wheeled vehicles and finger grasping are of very similar forms. There are two sets of equations of motions: one for the unconstrained vehicle or finger, the other for the ground (stationary) or the payload. These two sets of equations are coupled by the constraint force/torque that keep the vehicle on the ground with no wheel slippage or fingers on the object with no rotation about the local normal axis. Symbolically, these equations can be summarized in the following form:

$$\begin{array}{ll} (a): & M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u - J^T f \\ (b): & M_c\alpha_c + b_c + k_c = A^T f \\ (c): & Hf = 0 \\ (d): & v^+ = J\dot{\theta} \qquad v^- = Av_c = v^+ + H^T W \\ (e): & \alpha^+ = J\ddot{\theta} + \dot{J}\dot{\phi} \qquad \alpha^- = A\alpha_c + a = \alpha^+ + H^T\dot{W} + \dot{H}^T W. \end{array} \tag{2}$$

Eq.'s (2a)–(2b) are the equations of motion, f is the constraint force related to the vehicle or fingers via the Jacobian J^T, the null space of H, a full rank fat matrix, specifies the directions where motion at contact is allowed (therefore, no constraint force), v^+ and v^- are the velocity at two sides of the contacts, similarly, α^+ and α^- denote accelerations, W parameterizes the admissible velocity across the contact. The velocity constraint follows (2d), premultiplying by the annihilator of H^T, $\hat{H}^T$:

$$\hat{H}^T(J\dot{\theta} - Av_c) = 0. \tag{3}$$

In the single wheel case as in Figure 1, we have

$$\begin{array}{rcl} v_c & = & 0 \quad , \quad \dot{\theta} = \begin{bmatrix} \vec{\omega} \\ \vec{v} \end{bmatrix} \\ J & = & \begin{bmatrix} I & 0 \\ \ell\vec{z}\times & I \end{bmatrix} \quad , \quad H = [I \ , \ 0] \quad , \quad \hat{H}^T = [0 \ , \ I]. \end{array} \tag{4}$$

The velocity constraint (3) is then the same as (1).

2. *Conservation of Angular Momentum.*

   Consider a free floating multi-body system with no external torque (for example, a robot attached to a floating platform in space, or an astronaut unassisted by the jet pack, as shown in Figure 2.

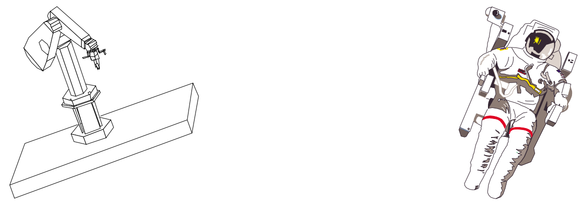

Figure 2: Examples of Free Floating Multi-body Systems

The equation of motion for such systems is give by [3]

$$\begin{bmatrix} M(q) & M_1(q) \\ M_1^T(q) & M_b(q) \end{bmatrix} \begin{bmatrix} \ddot{q} \\ \dot{\omega} \end{bmatrix} + \begin{bmatrix} C_{11}(q,\dot{q}) & C_{12}(q,\dot{q},\omega) \\ C_{21}(q,\dot{q},\omega) & C_{22}(q,\dot{q},\omega) \end{bmatrix} \begin{bmatrix} \dot{q} \\ \omega \end{bmatrix} = \begin{bmatrix} u \\ 0 \end{bmatrix} \tag{5}$$

where ω is the angular velocity of the multi-body system about the center of mass. By integrating the bottom portion in time, we obtain

$$M_1^T(q)\dot{q} + M_b(q)\omega = 0 \tag{6}$$

which is a non-integrable condition.

More generally, in a Lagrangian system, if a subset of the generalized coordinate q_u do not appear in the mass matrix $M(q)$, they are called the cyclic coordinate [4]. In this case, the Lagrangian equation associated with q_u is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_{u_i}}\right) = \frac{\partial L}{\partial q_{u_i}} = 0. \tag{7}$$

After integration, we obtain the conservation of generalized momentum condition associated with the cyclic coordinates. Eq. (6) is a special case of this situation.

3. *Underactuated Mechanical System.*

An underactuated mechanical system is one that does not have all of its degrees of freedom independently actuated. The non-integrable condition can arise in terms of velocity as we have seen above, or in terms of acceleration which cannot be integrated to a velocity condition. The latter case is called the *second order nonholonomic condition* [5].

First Order Condition. Consider a rigid spacecraft with less than three independent torquers.

$$I\dot{\omega} + \omega \times I\omega = Bu \tag{8}$$

where B is a full column rank matrix with rank less than three. Let $\hat{B}$ be the annihilator of B, i.e., $\hat{B}B = 0$. Then premultiplying (8) by $\hat{B}$ gives $\frac{d}{dt}(\hat{B}I\omega) = 0$. Assuming the initial velocity is zero, then we arrive at a non-integrable velocity constraint:

$$\hat{B}I\omega = 0.$$

Second Order Condition. Consider a robot with some of the joints unactuated. The general dynamic equation can be written as

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \begin{bmatrix} u \\ 0 \end{bmatrix}. \tag{9}$$

By premultiplying by $\hat{B} = [0 \ I]$ which annihilates the input vector, we obtain a condition involving the acceleration:

$$\hat{B}(M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q)) = 0. \tag{10}$$

It can be shown that this equation is integrable to a velocity condition, $h(q,\dot{q},t) = 0$, if and only if the following conditions hold [5]:

(a) the gravitational torque for the unactuated variables $g_u(q) = \hat{B}g(q)$ is a constant;

(b) the mass matrix $M(q)$ does not depend on the unactuated coordinate, $\hat{B}q$.

This implies that any earthbound robots with non-planar, articulated, underactuated degrees of freedom would satisfy a non-integrable second order constraint as $g_u(q)$ would not be constant.

The control problem associated with a nonholonomic system can be posed based on the kinematics alone (with an ideal dynamic controller assumed) or the full dynamical model.

In the kinematics case, nonholonomic conditions are linear in the velocity, v:

$$\Omega(q)v = 0. \tag{11}$$

Assuming that the rank of $\Omega(q)$ is constant over q, then (11) can be equivalently stated:

$$v = f(q)u \tag{12}$$

where the columns of $f(q)$ form a basis of the null space of $\Omega(q)$. Eq. (12) can be regarded as a control problem with u as the control variable and the configuration variable, q, as the state if $v = \dot{q}$. If v is non-integrable (as is the case for the angular velocity), there would be an additional kinematic equation $\dot{q} = h(q)v$ (such as the attitude kinematic equation); the control problem then becomes $\dot{q} = h(q)f(q)u$. Note that in either case, the right hand side of the differential equation is linear in u. Such systems are called *driftless systems* [6].

Solving the control problem associated with the kinematic equation (12) produces a feasible path. To actually follow the path, a real-time controller is needed to produce the required force or torque. This procedure of decomposing path planning and path following is common in industrial robot motion control. Alternatively, one can also consider the control of the full dynamical system directly. In other words, consider (2) for the rolling constraint case, or (5), (8) or (9) for the underactuated case, with u as the control input. In the rolling constraint case, the contact force may also needs to be controlled, similar to a robot performing a contact task. Otherwise, slippage or even lost of contact may result (e.g., witness occasional truck roll over on highway exit ramps). The dynamical equations also differ from the kinematic problem (12) in a fundamental way: a control-independent term, called the drift term, is present in the dynamics. In contrast to driftless systems, there is no known general global controllability condition for such systems. However, the presence of the drift term sometimes simplifies the problem by rendering the linearized system locally controllable.

There are also systems subject to nonholonomic inequality condition in the form of $\Omega(q)\dot{q} \leq 0$, but little analytic results are known for such systems.

This article will focus mainly on the kinematic control problem. In addition to the many research papers already published on this subject, excellent summary of the current state of research can be found in [7, 8, 9]. Specialized results for specific dynamic problems can be found in [10, 5, 11] for underactuated attitude and robot control, and pure rolling motion control problems. Elementary machinery in differential geometry that will be presented in this article can be found in several recently published texts in nonlinear control theory [6, 12, 13].

In the remainder of this article, we will address the following aspects on the kinematic control of a nonholonomic system:

1. *Determination of Nonholonomy.* Given a set of constraints, how does one classify them as holonomic or nonholonomic?

2. *Controllability.* Given a nonholonomic system, does there exist a path that connects an initial configuration to the desired final configuration?

3. *Path Planning.* Given a controllable nonholonomic system, how does one construct a path that connects an initial configuration to the desired final configuration?

4. *Stabilizability.* Given a nonholonomic system, can and how does one construct a stabilizing feedback controller?

5. *Output stabilizability.* Given a nonholonomic system, can and how does one construct a feedback controller that drives a specified output to the desired target while maintaining the boundedness of all the states?

We shall use a simple example to illustrate various concept and results throughout this section. Consider a unicycle with a fat wheel, i.e., it cannot fall (see Figure 3). For this

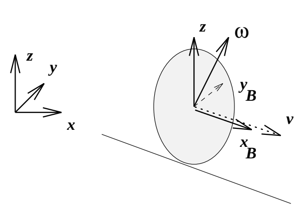

Figure 3: Unicycle Model and Coordinate Definition

system, there are four constraints:

$$\begin{aligned} \vec{x}_B \cdot \vec{\omega} &= 0 \\ \vec{v} - \vec{\omega} \times \ell \vec{z} &= 0. \end{aligned} \tag{13}$$

2 Test of Nonholonomy

As motivated in the previous section, consider a set of constraints in the following form:

$$\Omega(q)\dot{q} = 0 \tag{14}$$

where $q \in \mathcal{R}^n$ is the configuration variable, $\dot{q}$ is the corresponding velocity, and $\Omega(q) \in \mathcal{R}^{\ell \times n}$ is a matrix of constraints.

The complete integrability of the velocity condition in (14) means that $\Omega(q)$ is the Jacobian of some function, $h(q) \in \mathcal{R}^{\ell}$, i.e.,

$$\frac{\partial h}{\partial q} = \Omega(q). \tag{15}$$

In this case, (14) can be written as an equivalent holonomic condition: $h(q) = c$, where c is some constant vector. The condition (14) may be only partially integrable which means some of the rows of $\Omega(q)$, say, $\Omega_{k+1}, \ldots, \Omega_\ell$, satisfy

$$\frac{\partial h_i}{\partial q} = \Omega_i \quad , \quad i = k+1, \ldots, \ell. \tag{16}$$

for some scalar functions $h_i(q)$. Substituting (16) in (14), we have $\ell - k$ integrable constraints

$$\frac{\partial h_i}{\partial q}\dot{q} = 0. \tag{17}$$

which can be equivalently written as $h_i(q) = c_i$ for some constants c_i. If $\ell - k$ is the maximum number of such $h_i(q)$ functions, the rest k constraints are then nonholonomic.

To determine if the constraint (14) is integrable, we can apply the Frobenius Theorem. We first need some definitions:

Definition 1

1. *A* vector field *is a smooth mapping from the configuration space to the tangent space.*
2. *A* distribution *is the space generated by a collection of vector fields.*
3. *The* Lie Bracket *between two vector fields, f and g, is defined as*

$$[f, g] \triangleq \frac{\partial g}{\partial q} f(q) - \frac{\partial f}{\partial q} g(q).$$

4. *An* involutive distribution *is a distribution that is closed with respect to the Lie Bracket.*
5. *A distribution, Δ, consisting of vector fields in $\mathcal{R}^n$ with constant dimension m is* integrable *if there exist $n - m$ functions, $h_1, \ldots, h_{n-m}$ such that*

$$L_f h_i(q) \triangleq \frac{\partial h_i}{\partial q} \cdot f(q) = 0$$

for all $f \in \Delta$.

6. *The* involutive closure *of a distribution* Δ *is the smallest involutive distribution that contains* Δ*.*

The Frobenius Theorem can be simply stated:

Theorem 1 *A distribution is integrable if and only if it is involutive.*

To apply the Frobenius Theorem to (14), first observe that $\dot{q}$ must be within the null space of $\Omega(q)$ denoted by Δ. Suppose the constraints are independent throughout the configuration space, then the dimension of Δ is $n-\ell$; let a basis of Δ be $g_1(q),\ldots,g_{n-\ell}(q)$, i.e.,:

$$\Delta = \operatorname{span}\{g_1(q),\ldots,g_{n-\ell}(q)\}.$$

Let $\overline{\Delta}$ be the involutive closure of Δ. Suppose $\overline{\Delta}$ has constant dimension $n-\ell+k$. Since $\overline{\Delta}$ is involutive by definition, from the Frobenius Theorem, $\overline{\Delta}$ is integrable. This means that there exist functions h_i, $i=1,\ldots,\ell-k$, such that $\frac{\partial h_i}{\partial q} f = 0$ for all $f \in \overline{\Delta} \supset \Delta$. This implies that the flows of the system lie on a $n-\ell+k$ dimensional manifold given by h_i =constant, $i=1,\ldots,\ell-k$. Hence, among the ℓ constraints given by (14), $\ell-k$ is holonomic (obtained from the annihilator of $\overline{\Delta}$) and k is nonholonomic.

To illustrate the above discussion, consider the unicycle example presented at the end of Section 1. First write the constraints (13) in the same form as (11)

$$\begin{bmatrix} \vec{x}_B \cdot & 0 \\ \ell\vec{z}\times & I \end{bmatrix} \begin{bmatrix} \vec{\omega} \\ \vec{v} \end{bmatrix} = 0.$$

This implies that

$$\begin{bmatrix} \vec{\omega} \\ \vec{v} \end{bmatrix} \in \operatorname{span}\left\{ \begin{bmatrix} \vec{y}_B \\ \ell\vec{x}_B \end{bmatrix}, \begin{bmatrix} \vec{z} \\ 0 \end{bmatrix} \right\}.$$

Represent the top portion of each vector field in the body coordinate, $\vec{y}_B = [0,1,0]^T$, $\vec{z} = [0,0,1]^T$, and the bottom portion in the world coordinate, $\vec{x}_B = [\ell c_\theta, \ell s_\theta, 0]^T$, $c_\theta = \cos\theta$ and $s_\theta = \sin\theta$, θ is the steering angle, we have

$$\Delta = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ \ell c_\theta \\ \ell s_\theta \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}. \tag{18}$$

The involutive closure of Δ can be computed by taking repeated Lie Brackets:

$$\overline{\Delta} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ \ell c_\theta \\ \ell s_\theta \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\ell s_\theta \\ \ell c_\theta \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \ell c_\theta \\ \ell s_\theta \\ 0 \end{bmatrix} \right\}$$

which is of constant dimension four. The annihilator of $\overline{\Delta}$ is

$$\overline{\Delta}^{\perp} = \text{span}\{[1,0,0,0,0,0],[0,0,0,0,0,1]\}$$

From the Frobenius Theorem, the annihilator of $\overline{\Delta}$ is integrable. Indeed, the corresponding holonomic constraints are what one could have obtained by inspection

$$z = \text{ constant } \quad , \quad \psi = \text{ roll angle } = 0.$$

Eliminating the holonomic constraints result in the common form the kinematic equation for unicycle:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \ell c_\theta \\ \ell s_\theta \\ 0 \\ 1 \end{bmatrix} \omega_{y_B} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \omega_z. \tag{19}$$

Since the exact wheel rotational angle is frequently inconsequential, the ϕ equation is often omitted. Denoting ω_{y_B} and ω_z as inputs u_1 and u_2, the kinematic equation (19) can now be written as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \ell c_\theta \\ \ell s_\theta \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2. \tag{20}$$

We shall refer to the system described by this set of equation as the unicycle problem.

3 Nonholonomic Path Planning Problem

The nonholonomic path planning problem, also called the nonholonomic motion planning, involves finding a path connecting specified configurations that satisfies the nonholonomic condition as in (14). As discussed in Section 1, this problem can be written as an equivalent nonlinear control problem:

> *Given the following system*
>
> $$\dot{q} = f(q)u \; ; \; q \in \mathcal{R}^n \; , u \in \mathcal{R}^m \tag{21}$$
>
> *and initial and desired final configurations,* $q(0) = q_0$ *and* q_f*, find* $\underline{u} = \{u(t) : t \in [0,1]\}$ *such that the solution of (21) satisfies* $q(1) = q_f$.

Note the terminal time has been normalized to 1. In (21), $f(q)$ is a full rank matrix whose columns span the null space of $\Omega(q)$ in (14), and u parameterizes the degree of freedom in the velocity space. Note that by construction, $f(q)$ is necessarily a tall matrix.

3.1 Controllability

For $u = 0$, every q in $\mathcal{R}^n$ is an equilibrium. The linearized system about any equilibrium q^* is

$$\frac{d}{dt}(q - q^*) = f(q^*)u. \tag{22}$$

Since $f(q)$ is tall, this linear time invariant system is not controllable (the controllability matrix, $[f(q^*) \,|\, 0 \,|\, \ldots \,|\, 0]$, has maximum rank m). This is intuitively plausible, as the nonholonomic condition restricts the flows in the tangent space, the system can locally only move in directions compatible with the nonholonomic condition, contradicting the controllability requirement. However, the system may still be controllable globally.

A system such as (21) is called a system with no drift as the vector field does not contain a term only dependent on q. For such systems, the controllability of a nonlinear system can be ascertained through the following sufficient condition (sometimes called *Chow's Theorem* as it first appeared in [14]):

Theorem 2 *The system given by (21) is controllable if the involutive closure of the columns of $f(q)$ is of constant rank n for all q.*

The involutive closure of a set of vector fields is in general called the *Lie Algebra* generated by these vector fields. In the context of control systems where the vector fields are the columns of the input matrix $f(q)$, the Lie Algebra is called the *Control Lie Algebra*.

For systems with drift terms, the full rank condition is only sufficient for local accessibility. For a linear time invariant system, this condition simply reduces to the usual controllability rank condition. This theorem is non-constructive, however. The path planning problem basically deals with finding a specific control input to steer the system from a given initial condition to a given final condition, once the controllability rank condition is satisfied.

From the discussion in Section 2, it is clear that if the involutive closure of the null space of the constraints (14) (i.e., the Control Lie Algebra) is of constant rank for all configurations, the control system with the holonomic constraints removed is globally controllable. For the unicycle problem given by (19) or (20), the system is clearly globally controllable.

An alternate way to view (21) is to regard it as a nonlinear functional mapping of the input function $\underline{u}$ to the final state $q(1)$:

$$q(1) = F(q_0, \underline{u}). \tag{23}$$

Given $\underline{u}$, denote the solution of (21) by

$$q(t) = \phi_{\underline{u}}(t; q_0). \tag{24}$$

Then, $F(q_0, \underline{u}) = \phi_{\underline{u}}(1; q_0)$. In general, the analytic expression for F is impossible to obtain.

By definition, global controllability means that $F(q_0, \cdot)$ is an onto mapping for every q_0. For a given $\underline{u}$, $\nabla_{\underline{u}} F(q_0, \underline{u})$ corresponds to the system linearized about a trajectory $\underline{q} = \{q(t) : t \in [0, 1]\}$ which is generated by $\underline{u}$:

$$\delta\dot{q} = A(t)\delta q + B(t)\delta u \quad , \quad \delta q(0) = 0 \tag{25}$$

where, $A(t) \triangleq [\frac{\partial f}{\partial q^1}(q(t))u(t) \mid \cdots \mid \frac{\partial f}{\partial q^n}(q(t))u(t)]$ and $B(t) \triangleq f(q(t))$. Since $\delta q(0) = 0$, the solution to this equation is:

$$\delta q(1) = \int_0^1 \Phi(1,s)B(s)\delta u(s)\, ds \tag{26}$$

where Φ is the state transition matrix of the linearized system. It follows that

$$\left(\nabla_{\underline{u}} F(q_0, \underline{u})\right) \underline{v} = \int_0^1 \Phi(1,s)B(s)v(s)\, ds. \tag{27}$$

Controllability of the system in equation (25) implies that for any final state $\delta q(1) \in \mathcal{R}^n$, there exists a control δu which drives the linear system from $\delta q(0) = 0$ to $\delta q(1)$. This is equivalent to the operator $\nabla_{\underline{u}} F$ being onto (equivalently, the null space of the adjoint operator, $\left[\nabla_{\underline{u}} F\right]^*$, being zero). In the case that $\underline{u} = 0$, $\nabla_{\underline{u}} F$ reduces to the linear time invariant system (22). In this case, $\nabla_{\underline{u}} F$ obviously cannot be of full rank.

3.2 Path Planning Algorithms

3.2.1 Steering with Cyclic Input

In (21), since $f(q)$ is full rank for all q, there is a coordinate transformation so that $f(q)$ becomes $\begin{bmatrix} I \\ f_1(q) \end{bmatrix}$. In other words, the inputs are simply the velocities of m configuration variables. For example, in the unicycle problem described by (19), u_1 and u_2 are equal to $\dot{\theta}$ and $\dot{\phi}$. The subspace corresponding to these variables is called the *base space* (also called the *shape space*). A cyclic motion in the base space returns to the base variables to their starting point, but the configuration variables would have a net change (called the geometric phase) as shown in Figure 4. In the unicycle case, a cyclic motion in θ and ϕ results in the following net change in the x and y coordinates:

$$x(T) - x(0) = \int_0^T \cos\theta\dot{\phi}\, dt = \oint \cos\theta\, d\phi \quad ; \quad y(T) - x(0) = \int_0^T \sin\theta\dot{\phi}\, dt = \oint \sin\theta\, d\phi. \tag{28}$$

By the Green's Theorem, they can be written as surface integrals

$$x(T) - x(0) = \iint_S -\sin\theta\ d\theta\, d\phi \quad ; \quad y(T) - y(0) = \iint_S \cos\theta\ d\theta\, d\phi \tag{29}$$

where S is the surface enclosed by the closed contour in the $\phi - \theta$ space.

A general strategy for path planning would then consists of two steps: first drive the base variables to the desired final location, then appropriately choose a closed contour in the base space to achieve the desired change in the configuration variables without affecting the base variables. This basic idea has served as the basis of many path planning algorithms [15, 16, 17, 18].

To illustrate this procedure for path planning for the unicycle example, assume that the base variables, ϕ and θ, have reached their target values. We choose them to be sinusoids with integral frequencies, so that at $t = 1$, they return to their initial values:

$$u_1 = a_1 \cos(4\pi t) \quad , \quad u_2 = a_2 \cos(2\pi t). \tag{30}$$

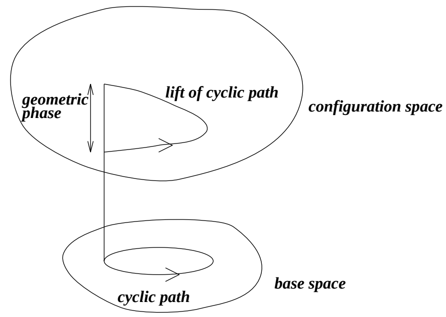

Figure 4: Geometric Phase

By direct integration, we have

$$\phi = \frac{a_1}{4\pi}\sin(4\pi t) \qquad , \qquad \theta = \frac{a_2}{2\pi}\sin(2\pi t). \tag{31}$$

For several values of a_1 and a_2, the closed contours in the $\phi - \theta$ plane given by (31) are as shown in Figure 3.2.1. The net changes in x and y over the period $[0, 1]$ are given by the surface integrals (29) over the area enclosed by the contours. To achieve the desired values for x and y, the two equations can be numerically solved for a_1 and a_2.

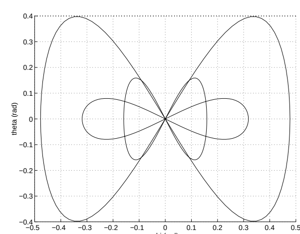

Figure 5: Closed Contour in Base Space

This procedure can also be carried directly in the time domain. For the chosen sinusoidal inputs, the changes in x and y are

$$\Delta x = \int_0^1 a_1 \cos(4\pi t)\cos\left(\frac{a_2}{2\pi}\sin(2\pi t)\right)\, dt \tag{32}$$

$$\Delta y = \int_0^1 a_1 \cos(4\pi t)\sin\left(\frac{a_2}{2\pi}\sin(2\pi t)\right)\, dt. \tag{33}$$

Using Fourier series expansion for even functions, we have

$$\cos\left(\frac{a_2}{2\pi}\sin(2\pi t)\right) = \sum_{k=0}^{\infty} \alpha_k \cos(2\pi k t)$$

$$\sin\left(\frac{a_2}{2\pi}\sin(2\pi t)\right) = \sum_{k=0}^{\infty} \beta_k \cos(2\pi t).$$

After the integration, we obtain

$$\Delta x = \frac{1}{2} a_1 \alpha_1 \quad , \quad \Delta y = \frac{1}{2} a_1 \beta_1. \tag{34}$$

Since α_1 and β_1 depend on a_2, given the desired motion in x and y, (34) results in a one-dimensional line search for a_2:

$$\frac{\Delta x}{\alpha_1(a_2)} - \frac{\Delta y}{\beta_1(a_2)} = 0. \tag{35}$$

Once a_2 is found (there may be multiple solutions), a_1 can be solved from (34).

The above procedure of using sinusoidal inputs for path planning can be generalized to systems in the following canonical form (written for systems with two inputs), called the *chain form*:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \vdots \\ \dot{q}_n \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ q_2 u_1 \\ q_3 u_1 \\ \vdots \\ q_{n-1} u_1. \end{bmatrix} \tag{36}$$

For example, the unicycle problem can be converted to the chain form by defining

$$\begin{aligned} q_1 &= \theta \\ q_2 &= c_\theta x + s_\theta y \\ q_3 &= s_\theta x - c_\theta y. \end{aligned}$$

Then

$$\begin{aligned} \dot{q}_1 &= u_1 \\ \dot{q}_2 &= -q_3 u_1 + \ell u_2 \\ \dot{q}_3 &= q_2 u_1. \end{aligned}$$

By defining the right hand side of the $\dot{q}_2$ equation as the new u_2, the system is now in the chain form.

For a general chain system, consider the following sinusoidal inputs,

$$u_1 = a \sin(2\pi t) \quad , \quad u_2 = b \cos(2\pi k t) \tag{37}$$

It follows that, for $i < k+2$, $q_i(t)$ consists of sinusoids with period 1; therefore, $q_i(1) = q_i(0)$. The net change in q_k can be computed to be

$$q_{k+2}(1) - q_{k+2}(0) = \left(\frac{a}{4\pi}\right)^k \frac{b}{k!}. \tag{38}$$

The parameters a and b can then be chosen so that q_{k+2} is driven to the desired value in $[0,1]$ without affecting all the states preceding it: q_i, $i < k+2$. A steering algorithm will then consists of the following steps

1. Drive q_1 and q_2 to the desired values.
2. For each q_{k+2}, $k = 1, \ldots, n-2$, drive q_{k+2} to its desired values by using the sinusoidal input (37) with a and b determined from (38).

Many systems can be converted to the chain form, e.g., kinematic car, space robot etc. However, there are also some systems that cannot be transformed to the chain form, e.g., a tractor with multiple trailers [19]. There has also been some recent generalizations as in [20]. In [17, 9], a general procedure is provided to transform a given system to the chain form.

3.2.2 Optimal Control

Another approach to nonholonomic path planning is optimal control. Consider the following two-input, three-state chain system (we have shown that the unicycle can be converted to this form):

$$\dot{q} = \begin{bmatrix} u_1 \\ u_2 \\ q_2 u_1 \end{bmatrix} \quad , \quad q(0) = q_0. \tag{39}$$

The inputs u_i are to be chosen to drive $q(t)$ from q_0 to $q(1) = 0$ while minimizing the input energy:

$$J = \int_0^1 \frac{1}{2} \|u(t)\|^2 \, dt.$$

The Hamiltonian associated with this optimal control problem is

$$H(q, u, \lambda) = \frac{1}{2} \|u\|^2 + \lambda^T f(q) u \tag{40}$$

where λ is the co-state vector. From the Maximum Principle, the optimal control can be found by minimizing H with respect to u:

$$u_1 = -(\lambda_1 + \lambda_3 q_2) \quad , \quad u_2 = -\lambda_2. \tag{41}$$

The co-state satisfies:

$$\dot{\lambda} = -\frac{\partial H}{\partial q} = \begin{bmatrix} 0 \\ -\lambda_3 u_1 \\ 0 \end{bmatrix}. \tag{42}$$

Differentiating the optimal control in (41), we obtain

$$\dot{u}_1 = c u_2 \quad , \quad \dot{u}_2 = -c u_1 \tag{43}$$

where c is a constant ($c = -\lambda_3$). This implies that u_1 and u_2 are sinusoids:

$$u_1(t) = -a \cos ct + u_1(0) \quad , \quad u_2(t) = a \sin ct + u_2(0). \tag{44}$$

Substituting in the equation of motion (39) and choosing $c = 2\pi$, we have

$$\begin{aligned} q_1(1) &= u_1(0) + q_1(0) \\ q_2(1) &= u_2(0) + q_2(0) \\ q_3(1) &= \frac{a^2}{4\pi} + \frac{u_1(0)a}{2\pi} + q_2(0)u_1(0) + \frac{u_1(0)u_2(0)}{2}. \end{aligned} \tag{45}$$

The requirement on the zero final state can be used to solve for the constants in the control:

$$\begin{aligned} u_1(0) &= -q_1(0) \\ u_2(0) &= -q_2(0) \\ a &= q_1(0) + sqrt q_1^2(0) + 2\pi q_1(0) q_2(0). \end{aligned} \tag{46}$$

If the expression within the square root is negative, then the constant c should be chosen as -2π to render it positive.

The optimization approach described above can be generalized to certain higher order systems, but, in general, the optimal control for nonholonomic systems is more complicated. One can also try finding an optimal solution numerically, this would in general entail the solution of a two-point boundary value problem. In [21, 22], a Ritz approximation approach is used where the input function is restricted to a finite dimensional functional space (e.g., finite Fourier basis):

$$u(t) = \sum_{k=0}^{N} \alpha_k \psi_k(t) \tag{47}$$

where ψ_k's are chosen to be independent orthonormal functions and α_k's are constant vectors parameterizing the input function. The minimum input energy criterion can then be combined with a final state penalty term, resulting in the following optimization criterion:

$$J = \gamma \left\| q_f - q(1) \right\|^2 + \int_0^1 \left\| u(t) \right\|^2 \, dt = \gamma \left\| q_f - q(1) \right\|^2 + \sum_{k=0}^{N} \left\| \alpha_k \right\|^2 . \tag{48}$$

The optimal α_k's can be solved numerically by using nonlinear programming. The penalty weighting γ can be increased to achieve the final target state while minimizing the control energy. The problem is not necessarily convex in general, consequently, as in any nonlinear programming problem, only local convergence can be asserted. This method is very similar to the approach in the next section, the main difference lies in that the control energy term is dropped in J. As a result of this modification, a stronger convergence condition can be established.

3.2.3 Path Space Iterative Approach

As shown in the beginning of Section 3, the differential equation governing the nonholonomic motion (21) can be written as a nonlinear operator relating an input function, $\underline{u}$, to a path, $\underline{q}$. By writing the final state error as

$$y = q_f - F(q_0, \underline{u}) \tag{49}$$

the path planning problem can be regarded as a nonlinear least square problem. Global controllability means that for any q_f there is at least one solution $\underline{u}$. Many numerical algorithms exist for the solution of this problem. In general, the solution involves lifting a path connecting the initial y to the desired $y = 0$ to the $\underline{u}$ space (see Figure 6). Let $\underline{u}(0)$ be the first guess of the input function and $y(0)$ be the corresponding final state error as given by (49). The goal is to iteratively modify $\underline{u}$ so y converges to 0 asymptotically. To this end,

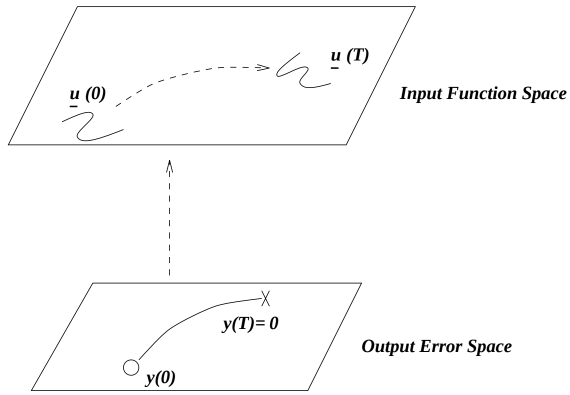

Figure 6: Path Planning by Lifting a Path in Output Error Space

choose a path in the error space connecting $y(0)$ to 0, call it $y_d(\tau)$ where τ is the iteration variable. The derivative of $y(\tau)$ is

$$\frac{dy}{d\tau} = -\nabla_{\underline{u}} F(q_0, \underline{u}) \frac{d\underline{u}}{d\tau}. \tag{50}$$

If $\nabla_{\underline{u}} F(q_0, \underline{u})$ is full rank, then we can choose the following update rule for $\underline{u}(\tau)$ to force y to follow y_d:

$$\frac{d\underline{u}}{d\tau} = -\left[\nabla_{\underline{u}} F(q_0, \underline{u})\right]^+ \left[-\alpha(y - y_d) + \frac{dy_d}{d\tau}\right] \tag{51}$$

where $\alpha > 0$ and $\left[\nabla_{\underline{u}} F(q_0, \underline{u})\right]^+$ denotes the Moore-Penrose pseudo-inverse of $\nabla_{\underline{u}} F(\underline{u})$. This is essentially the continuous version of Newton's Method. Eq. (51) is an initial value problem in $\underline{u}$ with a chosen $\underline{u}(0)$. With $\underline{u}$ discretized by a finite dimensional approximation (e.g., using Fourier basis as in (47)), it can be solved numerically by an ordinary differential equation solver.

As discussed in Section 3.1, the gradient of F, $\nabla_{\underline{u}} F(q_0, \underline{u})$ can be computed from the system (21) linearized about the path corresponding to $\underline{u}$. A sufficient condition for the convergence of the iterative algorithm (51) is that $\nabla_{\underline{u}} F(q_0, \underline{u}(\tau))$ for all τ, or equivalently, the time varying linearized system (25) generated by linearizing (21) about $\underline{u}(\tau)$ is controllable. For controllable systems without drift, it has been shown in [23] that this full rank condition is true generically (i.e., for almost all $\underline{u}$ in the C_∞ topology). The paths over which the time varying linearized system is uncontrollable (e.g., including all constant configurations) are called the abnormal extremals. In [24, 25], a sufficient condition is obtained to guaranteed the non-existence of such cases except for constant configuration paths. This has been generalized to the case of a unicycle where as long as the wheel velocity is not identically zero, the corresponding path is not an abnormal extremal.

In the cases where $\nabla_{\underline{u}} F(q_0, \underline{u})$ loses rank (causing the algorithm to possibly get stuck), [26] observed that a *generic loop* (see Figure 7) can be appended to the singular control causing the composite control to be nonsingular and thus allowing the algorithm to continue its progress toward a solution. A generic loop can be described as follows. For some small time interval $[0, T/2]$, generate a nonsingular control $\underline{v_a}$ then let $\underline{v}$ be the control on $[0, T]$ consisting of $\underline{v_a}$ on $[0, T/2]$ and $-\underline{v_a}$ on $[T/2, T]$. Since nonholonomic systems have no drift term, it follows that the system makes a "loop" starting at $q(1) = F(q_0, \underline{u})$ and ending once again at the same point. Appending $\underline{v}$ to $\underline{u}$ and renormalizing the time interval to $[0, 1]$ yields a nonsingular control which does not change y. The algorithm is therefore guaranteed to converge to any arbitrary neighborhood of the desired final configuration.

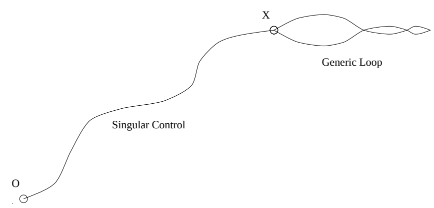

Figure 7: Generic Loop

This algorithm has been extended to include inequality constraints such as joint limits, collision avoidance etc. [27]. This is done by using an exterior penalty function approach [28]. The state inequality constraints given by

$$c(\underline{q}) \leq 0 \tag{52}$$

where $\underline{q}$ is the complete path in the configuration space, $c(\cdot)$ is a vector and the inequality is interpreted in the component-wise sense. The state trajectory, $\underline{q}$, can be related to the input function $\underline{u}$ through a nonlinear operator (which is typically impossible to find analytically)

$$\underline{q} = \mathcal{F}(q_0, \underline{u}). \tag{53}$$

The inequality constraint (52) can then be expressed in terms of $\underline{u}$:

$$c(\mathcal{F}(q_0, \underline{u})) \leq 0. \tag{54}$$

Inequality constraints in optimization problems are typically handled through penalty functions [28]. There are two types: interior and exterior penalty functions. Interior penalty function sets up barriers at the boundary of the inequality constraints. As the height of the barrier increases, the corresponding path becomes closer to being feasible. If the optimization problem — in our case, the feasible path problem — can be solved for each finite barrier, then convergence to the optimal solution is assured as the barrier height tends to infinity. In the exterior penalty function approach, the i^{th} inequality constraint is converted to an equality constraint by using an exterior penalty functions:

$$z_i(\underline{u}) = \gamma_i \sum_{j=1}^{N} g(c_i(\mathcal{F}_j(q_0, \underline{u}))) \tag{55}$$

where $\gamma_i > 0$, c_i is the i^{th} constraint, $\mathcal{F}_j$ denotes the j^{th} discretized time point where the constraint is checked, g is a continuous scalar function with the property that g is equal to zero when c_i is less than or equal to zero and is greater than zero and monotonic when c_i is greater than zero. The same iterative approach presented for the equality-only case can now be applied to the composite constraint vector:

$$\psi(\underline{u}) = \begin{bmatrix} y(\underline{u}) \\ z(\underline{u}) \end{bmatrix}. \tag{56}$$

For a certain class of convex polyhedral constraints, the generic full rank condition for the augmented problem still holds. This approach has been successfully applied to many complex examples such as cars with multiple trailers subject to a variety of collision avoidance and joint limits constraints [29].

3.2.4 Other Methods

There has also been work by using piecewise constant input by Lafferriere and Sussmann [30, 31] for nilpotent systems. This was further extended by Sussmann and Liu to approximate any path, which may violate the nonholonomic constraint, arbitrarily closely by a nonholonomic one [32]. The resulting path may contain may forward and backup maneuver, however. In [33], this idea is coupled with a general path planner (which includes collision avoidance but not nonholonomic constraints) but added with a refinement step that smoothes the path to produce an acceptable final path.

The path planning problem can also be tackled directly by using search algorithms without first posing it as a control problem. Work along this line include [34, 35, 36]. Due to the nature of the approach, inequality constraints such as joint limits, obstacles in the work space can be directly considered. However, the computation load associated with search methods has so far limited applications to only relatively simple systems, such as a truck with one trailer.

4 Stabilization

4.1 State Stabilization

Stabilizability means the existence of a feedback controller that will render the closed loop system asymptotically stable about an equilibrium point. For linear systems, controllability implies stabilizability. It would be of great value if this were true for special classes of nonlinear systems such as driftless systems such as nonholonomic systems considered in this paper (where controllability can be checked through a rank condition on the Control Lie Algebra). It was shown by [37, 38] that this assertion is not true in general. For a general nonlinear system $\dot{q} = f_0(q, u)$ with equilibrium at q_0, $f_0(q_0, 0) = 0$, and $f_0(\cdot, \cdot)$ continuous in a neighborhood of $(q_0, 0)$, a necessary condition for the existence of a continuous time invariant control law which renders $(q_0, 0)$ asymptotically stable is that f maps any neighborhood of $(q_0, 0)$ to a neighborhood of 0. For a nonholonomic system described by (21), $f_0(q, u) = f(q)u$. Then the range of $\{f_0(q, u) : (q, u) \text{ in a neighborhood of } (q_0, 0)\}$ is equal to the span of the columns of $f(q)$ which is of dimension m (number of inputs). Since a neighborhood about the zero state is n dimensional, the necessary condition above is not satisfied unless $m \geq n$.

There are two approaches to deal with the lack of continuous time invariant feedback. The first is to relax on the continuity requirement to allow for piecewise smooth control law; the second is to relax on the time invariance requirement and allow for a time varying feedback.

In either approach, an obvious starting point is to begin with an initial feasible path obtained by using any one of the open loop methods discussed in Section 3 and then apply a feedback to stabilize the system around the path. This can be done using a switching type of controller as in [11, 39] or a time varying controller as in [40, 41, 42]. This problem is essentially a feedback path following problem. As the nonholonomic model is highly idealized (for example, in a real car, the wheels can slip), using feedback to maintain the system on the planned path is indispensable.

Given an initial feasible path, if the nonlinear kinematic model linearized about the path is controllable, which is almost always true as mentioned in Section 3.2.3, a time varying stabilizing controller can be constructed by using standard techniques, see, for example, [43]. The resulting system will then be locally asymptotically stable.

Consider the unicycle as an example. Suppose an open loop trajectory, $\{(x^*(t), y^*(t), \theta^*(t), t \in [0,1]\}$, and the corresponding input, $\{u_1^*(t), u_2^*(t), t \in [0,1]\}$, are already generated by using any of the methods discussed in Section 3. The system equation can be linearized about this path:

$$\begin{aligned} \delta\dot{x} &= -(\sin(\theta^*(t))\, u_1^*(t))\, \delta\theta + \cos(\theta^*(t))\, \delta u_1 \\ \delta\dot{y} &= (\cos(\theta^*(t)) u_1^*(t))\, \delta\theta + \sin(\theta^*(t))\, \delta u_1 \\ \delta\dot{\theta} &= \delta u_2. \end{aligned} \tag{57}$$

This is a linear time varying system. It can easily verified that as long as u_1^* is not identically zero, the system is controllable. One can then construct a (time-varying) stabilizing feedback to keep the system on the planned open loop path.

Stabilizing control laws can also be directly constructed without first finding a feasible open loop path. In [44], it was shown that all nonholonomic systems can be feedback stabilized with a smooth periodic controller. For specific classes of systems, such as mobile robots in [40, 39] or, more generally, the so-called power systems as in [41], explicit constructive procedures for such controllers have been demonstrated. It is also possible to find piecewise smooth time-invariant controllers based on output stabilization (see next section for an example) or transformation of the configuration space [39].

We will again use the unicycle example to illustrate the basic idea of constructing a time varying stabilizing feedback by using a time dependent coordinate transformation so that the equation of motion contains a time varying drift term. Define a new variable z by

$$z = \theta + k(t, x, y) \tag{58}$$

where k is a function that will be specified later. Differentiating z, we get

$$\dot{z} = v \triangleq u_2 + \frac{\partial k}{\partial t} + (\frac{\partial k}{\partial x} \cos\theta + \frac{\partial k}{\partial y} \sin\theta) u_1.$$

Consider a quadratic Lyapunov function candidate $V = \frac{1}{2}(x^2 + y^2 + z^2)$, the derivative along the solution trajectory is

$$\dot{V} = (x\cos\theta + y\sin\theta)u_1 + zv.$$

By choosing

$$u_1 = -\alpha_1(x\cos\theta + y\sin\theta) \quad , \quad v = -\alpha_2 z \qquad \alpha_1, \alpha_2 > 0, \tag{59}$$

which means

$$u_2 = -\frac{\partial k}{\partial t} - (\frac{\partial k}{\partial x}\cos\theta + \frac{\partial k}{\partial y}\sin\theta)u_1 - \alpha_2(\theta + k(t,x,y)),$$

we obtain a negative semidefinite $\dot{V} = -\alpha_1(x\cos\theta + y\sin\theta)^2 - \alpha_2 z^2$. This implies that, as $t \to \infty$, $z \to 0$ and $x\cos\theta + y\sin\theta \to 0$. Substituting in the definition of z, we get $\theta(t) \to -k(t, x(t), y(t))$. From the other asymptotic condition, $\theta(t)$ also converges to $-\tan^{-1}(\frac{x(t)}{y(t)})$. As $\dot{x}$ and $\dot{y}$ asymptotically vanishes, $x(t)$ and $y(t)$, and therefore, $\theta(t)$, tend to a constant. Equating the two asymptotic expressions for $\theta(t)$, we conclude that $k(t, x(t), y(t))$ converges to a constant. By suitably choosing $k(t,x,y)$, e.g., $k(t,x,y) = (x^2 + y^2)\sin(t)$, the only condition under which $k(t,x,y)$ can converge to a constant is that $x^2 + y^2$ converges to zero, which in turn implies that $\theta(t) \to 0$. In contrast to the indirect approach (i.e., use a linear time varying control law to stabilize system about a planned open loop path), this control law is globally stabilizing.

4.2 Output Stabilization

In certain cases, it may only be necessary to control the state to a certain manifold rather than to a particular configuration. For example, in the case of a robot manipulator on a free floating mobile base, it may only be necessary to just control the tip of the manipulator so it can perform useful task. In this case, a smooth output stabilizing controller can frequently be found.

Suppose the output of interest is

$$y = g(q) \ , \ y \in \mathcal{R}^p \tag{60}$$

and $p < n$. at a particular configuration, q,

$$\dot{y} = \nabla_q g(q)\, f(q)\, u. \tag{61}$$

Define $K(q) = \nabla_q g(q)\, f(q)$. If $K(q)$ is onto, i.e., $p \leq m$ and $K(q)$ is full rank, then the system is locally output controllable (there is a u that can move y arbitrarily within a small enough ball) though it is not locally state controllable.

The output stabilization problem involves finding a feedback controller u (possibly dependent on the full state) to drive y to a setpoint, y_d. Provided that $K(q)$ is of full row rank, an output stabilizing controller can be easily found:

$$u = -QK^T(q)(y - y_d) \quad , \quad Q > 0. \tag{62}$$

Therefore, y is governed by

$$\dot{y} = -K(q)QK^T(q)(y - y_d). \tag{63}$$

Under the full row rank assumption on $K(q)$, $K(q)QK^T(q)$ is positive definite which implies that y converges to y_d asymptotically. In general, either y converges to y_d or q converges to a singular configuration of $K(q)$ (where $K(q)$ loses rank).

We will again use the unicycle problem as an illustration. Suppose the output of interest is (x, θ) and the goal is to drive (x, θ) to (x_d, θ_d) where θ_d is not a multiple of $\frac{\pi}{2}$. By choosing the control law

$$u_1 = -\alpha_1 \cos\theta (x - x_d) \quad , \quad u_2 = -\alpha_2(\theta - \theta_d). \tag{64}$$

The closed loop system for the output is

$$\dot{x} = -\alpha_1(\cos^2\theta)(x - x_d) \quad , \quad \dot{\theta} = -\alpha_2(\theta - \theta_d). \tag{65}$$

The closed loop system contains a singularity at $\theta = \frac{\pi}{2}$, but if $\theta_d \neq \frac{\pi}{2}$, this singularity will not be attractive. The output stabilization of (x, θ) can be concatenated with other output stabilizing controllers with other choices of outputs to obtain full state stabilization. For example, once x is driven to zero, θ can be independently driven to zero (with $u_1 = 0$), and, finally, y can be driven to zero without affecting x and θ. These stages can be combined together as a piecewise smooth state stabilizing feedback controller.

Consider a space robot on a platform as another example. Suppose the output of interest is the end effector coordinate, y. The singular configurations in this case are called the dynamic singularities. The output velocity is related to the joint velocity and center of mass angular velocity by the kinematic Jacobians:

$$\dot{y} = J(q)\dot{q} + J_b(q)\omega.$$

As discussed in Section 1, the nonholonomic nature of the problem follows from the conservation of the angular momentum (6):

$$M_1^T(q)\dot{q} + M_b(q)\omega = 0.$$

Eliminating ω, we obtain

$$\dot{y} = (J(q) - J_b M_b^{-1} M_1^T(q))\dot{q}.$$

The effective Jacobian, $K = J(q) - J_b M_b^{-1} M_1^T(q)$, sometimes called the dynamic Jacobian, now contains inertia parameters (hence the modifier "dynamic") in contrast to a terrestrial robot Jacobians which only depend on the kinematic parameters. If the dimension of q is at least as large as the dimension of y, the output can be effectively controlled provided that the dynamic Jacobian does not lose rank (i.e., q is away from the dynamic singularities).

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