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Discontinuous control of nonholonomic systems

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Abstract

The problem of asymptotic convergence for a class of nonholonomic control systems via *discontinuous* control is addressed and solved from a new point of view. It is shown that control laws, which ensures asymptotic (exponential) convergence of the closed-loop system, can be easily designed if the system is described in *proper* coordinates. In such coordinates, the system is discontinuous. Hence, the problem of local asymptotic stabilization for a class of discontinuous nonholonomic control systems is dealt with and a general procedure to transform a continuous system into a discontinuous one is presented. Moreover, a general strategy to design discontinuous control laws, yielding asymptotic convergence, for a class of nonholonomic control systems is proposed. The enclosed simulation results show the effectiveness of the method.

Keywords: Nonholonomic systems; σ process; Discontinuous control

1. Introduction

The problem of feedback stabilization of nonlinear systems has occupied a central role in the nonlinear systems literature for at least three decades. One of the most challenging topics in this area is the design of local (or global) stabilizing control laws for nonholonomic systems with more degrees of freedom than controls. As pointed out in an early paper of Brockett [9], such control systems cannot be stabilized by continuously differentiable, time invariant, state feedback control laws. Hence, a considerable effort has been expended in order to find continuous, time varying control laws [13, 16, 17], discontinuous ones [8, 10, 11] and middle strategies (discontinuous and time varying) [14, 15, 19].

However, most of the existing approaches, though extremely sophisticated and elegant, suffer from several drawbacks. Time varying control laws are, most of the time, extremely complex, their design is far from intuitive, there is no unique idea underlying their synthesis and only for a special class of nonholonomic systems a general strategy is available [16]. Moreover, as pointed out in [15] any C^1, periodic, state feedback control law is unable to exponentially stabilize the closed-loop system. Thus, time varying control laws produce very slow convergence and, what is worse, are intrinsically oscillating. The oscillatory behavior shown by many controlled nonholonomic systems (see the results in [11, 13, 17]) is not intrinsic to the system and is not even necessary to *move* the system from an initial configuration to the desired final one. Thus, we deduce that such oscillatory behavior is a byproduct of the stabilization procedure and can be avoided using different approaches or control laws. On the other hand, discontinuous control laws are able to provide exponential stability. However, their design is not simple and only for a particular class of systems, namely those feedback equivalent to chained systems, exponentially stabilizing, discontinuous control laws are available [11, 19].

In the present work we study the class of nonholonomic control systems described by equations of the

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form

$$\dot{x} = g(x)u, \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are state variable and control input, respectively, and we restrict our attention to discontinuous control laws, i.e. control laws modeled by equations of the form

$$u = \alpha(x),$$

where $\alpha : \mathbb{R}^n \to \mathbb{R}^m$ is a discontinuous function of its arguments [1]. Discontinuous control laws have already been dealt with, from a different point of view, in [11, 12] whereas some features of the proposed approach have been presented in [2–5].

2. On the stabilization of discontinuous nonholonomic systems

This section contains our main results, namely a sufficient condition of stabilizability for systems described by equations of the form (1) with $n > m$. For, we establish now a preliminary result, pointing out the geometrical properties of a class of nonlinear nonholonomic systems.

Lemma 1. *Consider the system*

$$\begin{aligned} \dot{x}_1 &= g_{11}(x_2)u_1, \\ \dot{x}_2 &= g_{21}(x_1,x_2)u_1 + g_{22}(x_1,x_2)u_2, \end{aligned} \tag{2}$$

with $x_1 \in \mathbb{R}^p$, $x_2 \in \mathbb{R}^{n-p}$, $x = \mathrm{col}(x_1,x_2) \in \mathbb{R}^n$, $u_1 \in \mathbb{R}^p$, $u_2 \in \mathbb{R}^m$, $m + p < n$,

$$g(x) = \begin{bmatrix} g_{11}(x_2) & 0 \\ g_{21}(x_1,x_2) & g_{22}(x_1,x_2) \end{bmatrix}$$

of constant rank in a neighborhood U_0 *of* $x = 0$ *(*$g_{ij}(x_1,x_2)$ *are matrices of appropriate dimensions). Let* U *be an open and dense set such that* $U \subset U_0$ *and* $\{x \in U_0 \mid x_1 = 0\} \not\subset U$.

Assume the following.

(i) *The matrix function* $g_{21}(x_1,x_2)$ *is smooth in* U *and the matrix functions* $g_{11}(x_1,x_2)$ *and* $g_{22}(x_1,x_2)$ *are smooth in* U_0.

(ii) *Let* $u_1 = u_1(x_1,x_2)$ *be a smooth mapping such that*

$$u_1(0,x_2) = 0, \tag{3}$$

for all x_2, *and such that the vector field* $g_{21}(x_1,x_2)u_1(x_1,x_2)$ *is smooth in* U_0.

Then, for every u_2, *the* $n - p$ *dimensional manifold* $\mathcal{M} = \{x \in \bar{U} : x_1 = 0\}$ *is invariant for the system*

$$\begin{aligned} \dot{x}_1 &= g_{11}(x_1,x_2)u_1(x_1,x_2), \\ \dot{x}_2 &= g_{21}(x_1,x_2)u_1(x_1,x_2) + g_{22}(x_1,x_2)u_2. \end{aligned} \tag{4}$$

Proof. To prove the claim it suffices to show that every trajectory of the closed-loop system (4) starting at $x(0) = (x_1(0),x_2(0)) = (0,x_{20})$ yields $x_1(t) \equiv 0$ for all $t \geqslant 0$. This is indeed true as, by Eq. (3), the term $g_{11}(x_1,x_2)u_1(x_1,x_2)$ is zero, at $x_1 = 0$, for every x_2. □

Remark 1. It is interesting to consider the well defined restriction of the dynamics of system (4) to the invariant manifold $\mathcal{M}$, described, if $g_{22}(x_1,x_2)$ and $g_{21}(x_1,x_2)u_1(x_1,x_2)$ are smooth in U_0, by equations of the form

$$\dot{x}_2 = \lim_{x_1 \to 0} (g_{21}(x_1,x_2)u_1(x_1,x_2)) + g_{22}(0,x_2)u_2.$$

Such equations play a fundamental role in all the following developments.

Remark 2. System (2) is not really a special nonholonomic system. In fact, as discussed in [8]; under mild hypotheses and with a proper choice of coordinates, it is always possible to write the kinematic equations of a nonholonomic system in the form of equations (2) with

$$g_{11}(x_1) = I_p, \qquad g_{21} = \begin{bmatrix} 0 \\ \bigstar(x_1,x_2) \end{bmatrix},$$

$$g_{22} = \begin{bmatrix} I_m \\ \bigstar(x_1,x_2) \end{bmatrix},$$

where $\bigstar(x_1,x_2)$ denotes a generic function of x_1 and x_2.

The existence of an invariant manifold for the closed-loop system (4) has a special relevance in deriving sufficient conditions for stabilizability, as discussed in the following statement.

Theorem 1. *Consider the system* (2). *Let* U_0 *be a neighborhood of the origin of* $\mathbb{R}^n$ *and let* U *be an open and dense set such that* $U \subset U_0$ *and* $\{x \in U_0 \mid x_1 = 0\} \not\subset U$.

[1] It must be noticed that control laws which are not defined at $x = 0$, i.e. are unbounded at $x = 0$ are allowed.

Suppose the following.

(i) *The matrix functions $g_{11}(x_1,x_2)$ and $g_{22}(x_1,x_2)$ have smooth entries in U_0.*

(ii) *The matrix function $g_{21}(x_1,x_2)$ has smooth entries in U.*

(iii) *The matrix function $g_{22}(x_1,x_2)$ is such that, for all i and j,*

$$\partial(g_{22}(x_1,x_2))_{ij}/\partial x_1 \equiv 0,$$

i.e. $g_{22}(x_1,x_2)$ is a function of x_2 only, say $\tilde{g}_{22}(x_2)$.

(iv) *There exists a smooth vector function $u_1(x_1,x_2)$, zero for $x_1 = 0$ and for all x_2, i.e. $u_1(0,x_2) = 0$, such that*

($\mathrm{iv_A}$) $-\infty < x_1^{\mathrm{T}} X g_{11}(x_1,x_2) u_1(x_1,x_2) < 0$, *for some positive-definite matrix X and for all nonzero x_1 in U_0;*

($\mathrm{iv_B}$) *the vector function $g_{21}(x_1,x_2)u_1(x_1,x_2)$ is smooth in U_0 and fulfills, for all i,*

$$\partial(g_{21}(x_1,x_2)u_1(x_1,x_2))_i/\partial x_1 \equiv 0,$$

i.e. $(g_{21}(x_1,x_2)u_1(x_1,x_2))$ is a function of x_2 only, say $\tilde{f}_2(x_2)$, and $\tilde{f}_2(0) = 0$.

(v) *There exists a smooth function $u_2(x_2)$ which renders the equilibrium $x_2 = 0$ of the system*

$$\dot{x}_2 = \tilde{f}_2(x_2) + \tilde{g}_{22}(x_2)u_2(x_2) \tag{5}$$

locally asymptotically stable.

Then, the smooth control law

$$u = u(x_1,x_2) = \begin{bmatrix} u_1(x_1,x_2) \\ u_2(x_2) \end{bmatrix}$$

locally asymptotically stabilizes the system (2).

Proof. First of all observe that the closed-loop system is smooth in U_0, hence, to show local asymptotic stability we can use standard Lyapunov theory.

By the local inverse Lyapunov theorem [6, Appendix A], there exists a smooth positive-definite function $V(x_2)$ such that

$$\dot{V} = V_{x_2}(\tilde{f}_2(x_2) + \tilde{g}_{22}(x_2)u_2(x_2)) < 0,$$

for all nonzero x_2. Consider now the smooth positive-definite function

$$W(x_1,x_2) = \tfrac{1}{2}x_1^{\mathrm{T}} X x_1 + V(x_2)$$

and observe that

$$\dot{W} = x_1^{\mathrm{T}} X g_{11}(x_1,x_2)u_1(x_1,x_2) + V_{x_2}(\tilde{f}_2(x_2) + \tilde{g}_{22}(x_2)u_2(x_2))$$

is negative definite, for all nonzero (x_1,x_2), in a neighborhood of the origin. Hence the proof is complete. □

As should be evident from Theorem 1, the existence of the invariant manifold $x_1 = 0$ for the closed-loop system (4) allows to solve the asymptotic stabilization problem in two successive steps. Hypothesis (iv) determines the component u_1 of the control law; whereas the component u_2 must be chosen to fulfill hypothesis (v). Observe that the choice of u_1 is crucial, as the existence of a smooth function $u_2(x_2)$ fulfilling hypothesis (v) depends on such a choice.

Remark 3. Very often, the control law $u_2 = u_2(x_2)$ can be designed on the basis of the linear system

$$\dot{\tilde{x}}_2 = \left[\frac{\partial \tilde{f}(x_2)}{\partial x_2}\right]_{x_2=0} \tilde{x}_2 + \tilde{g}_{22}(0)v.$$

Note that if the function $\tilde{f}_2(x_2)$, defined in hypothesis (iv) of Theorem 1, is identically zero then, by Brockett theorem [9], there exists no smooth control law $u_2(x_2)$ which renders the system (5) locally asymptotically stable. However, if $g_{21}(x_1,x_2)$ and $u_1(x_1,x_2)$ are continuous at $x_1 = 0$ then certainly $g_{21}(x_1,x_2)u_1(x_1,x_2)$ depends also on x_1, i.e. the dependence on x_1 does not cancel out in the product. Since in Theorem 1 $u_1(x_1,x_2)$ was chosen to be a smooth function with $u_1(0,x_2) = 0$, it follows that

$$\lim_{x_1 \to 0} g_{21}(x_1,x_2) = \pm\infty.$$

Hence, in this paper, we consider systems described by equations of the form (2) with $g_{21}(x_1,x_2)$ such that there exist a smooth function $\psi(x_1,x_2)$, fulfilling $\psi(0,x_2) = 0$, and a smooth matrix $\phi(x_1,x_2)$, fulfilling $\phi(0,x_2) \neq 0_{(n-p)\times p}$, such that

$$g_{21}(x_1,x_2) = \phi(x_1,x_2)/\psi(x_1,x_2),$$

i.e. system (2) is discontinuous (not defined) for $x_1 = 0$.

3. The σ process

As a consequence of the results of the previous section, we conclude that a nonholonomic system described by equations of the form (2) admits a (local) smooth stabilizer only if it is discontinuous (not defined) on the hyperplane $x_1 = 0$. As natural systems are almost never discontinuous, in this section we address the problem of transforming a smooth system into a discontinuous one. Such a task can be performed applying to the original (smooth) system a discontinuous coordinates transformation, i.e. a σ process [1].

The σ process has been introduced in the theory of dynamical systems to study the behavior of a system near a given point. Mainly, it consists of a rational (discontinuous) coordinates transformation, possessing the property of *increasing the resolution* around a given point. A possible example of σ process is the rational transformation in the plane

$$w = y^\alpha / x^\beta, \qquad z = x,$$

with $\alpha > 0$, $\beta > 0$ and defined for $x \neq 0$.

Consider a nonholonomic system described by equations of the form (2). Assume that $g_{11}(x_2) = I_p$ and that the matrices $g_{21}(x_1,x_2)$ and $g_{22}(x_1,x_2)$ have smooth entries. Consider, moreover, a coordinates transformation (σ process) described by equations of the form

$$\begin{aligned} \xi_1 &= x_1, \\ \xi_2 &= \Phi_2(x_1,x_2)/\sigma(x_1), \end{aligned} \tag{6}$$

where $\Phi_2(x_1,x_2)$ is a smooth mapping in a neighborhood U^0 of $x = 0$ and $\sigma(x_1)$ is a smooth function in a neighborhood W of $x_1 = 0$ and is such that $\sigma(x_1) = 0$.

Remark 4. The transformation (6) defines a σ process if $\Phi_2(0,x_2) \neq 0_{n-p}$.

The transformed system is described, in the new coordinates, always by equations of the form (2); but, in general, the matrix $g_{21}(\xi_1,\xi_2)$ is not defined at $\xi_1 = 0$. Thus, we conclude that the σ process allows to *map* the space of smooth systems into the space of discontinuous ones.

4. An algorithm to design discontinuous stabilizer for nonholonomic systems

In the present section, using the results developed, we propose a procedure to design discontinuous control laws for smooth nonholonomic systems described by equations of the form (2).

The procedure is composed of the following steps:

(I) Transform the given smooth nonholonomic system, by means of a σ process described by equations of the form (6), into a discontinuous system.

(II) Check if the discontinuous system admits a smooth control law yielding local asymptotic stability. In case of positive answer proceed to step (III), otherwise return to step (I) and apply a different σ process.

(III) Build a smooth stabilizer for the transformed system.

(IV) Apply the inverse σ process to the obtained stabilizer to build a discontinuous control law for the original system.

Remark 5. The crucial points of the algorithm are the selection of the σ process (step (I)) and the design of the smooth asymptotically stabilizing control law for the transformed system (step (III)). In particular, step (III) can be easily solved for low dimensional systems; whereas there is no constructive or systematic way to perform step (I) successfully.

Remark 6. The discontinuous control law resulting from the above algorithm is not, in general, a discontinuous stabilizer for the original smooth nonholonomic system. In fact, asymptotic stability of the transformed system (with state ξ) does not imply asymptotic stability of the original system (with state x) as the inverse of the coordinates transformation (6) does not map neighborhood of $\xi = 0$ into neighborhood of $x = 0$. As a consequence, asymptotic (exponential) stability of the closed-loop system with state ξ implies only asymptotic (exponential) convergence in an open and dense set of the closed-loop system with state x; see [3, 5] for further detail.

5. Some examples

In the present section we use the theory developed to design discontinuous control laws for some interesting and prototype nonholonomic systems. Further examples can be found in [2–5].

5.1. A knife edge

Consider the control of a knife edge moving in point contact on a planar surface [7]. Let x_1 and x_2 denote the (x, y) coordinates of the point of contact of the knife edge on the plane and let x_3 denote the heading angle of the knife edge, measured from the x axis. Then, the kinematic equations of motion are (all constants are set to unity) [7]

$$\begin{aligned} \dot{x}_1 &= v_1 \cos x_3, \\ \dot{x}_2 &= v_1 \sin x_3, \\ \dot{x}_3 &= v_2, \end{aligned} \tag{7}$$

where v_1 denotes the velocity in the direction defined by the heading angle and v_2 the angular velocity about the vertical axis through the point of contact. Define $u_1 = v_1 \cos x_3$ and $u_2 = v_2$.

Execute now the proposed algorithm.

(I) Apply the σ process

$$\begin{aligned} \xi_1 &= x_1, \\ \xi_2 &= x_2, \\ \xi_3 &= x_3/x_1. \end{aligned}$$

The transformed system is described by the equations

$$\begin{aligned} \dot{\xi}_1 &= u_1, \\ \dot{\xi}_2 &= \frac{\tan \xi_3 - \xi_2}{\xi_1} u_1, \\ \dot{\xi}_3 &= u_2. \end{aligned} \tag{8}$$

(II) and (III) The control law

$$u = \begin{bmatrix} -k\xi_1 \\ p_2\xi_2 + p_3\xi_3 \end{bmatrix} \tag{9}$$

with $k > 0$ and p_2 and p_3 such that the matrix

$$A = \begin{bmatrix} k & -k \\ p_2 & p_3 \end{bmatrix}$$

has all its eigenvalues with negative real part [2], locally asymptotically stabilizes system (8).

(IV) In the original coordinates system the control law is described by equations of the form

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -k(x_1/\cos x_3) \\ p_2 x_2 + p_3(x_3/x_1) \end{bmatrix}. \tag{10}$$

Remark 7. From local asymptotical (exponential) stability of the closed-loop system (8)–(9) we infer that the discontinuous control law (10) is well defined and bounded, for all $t \geqslant 0$, along the trajectories of the closed-loop system (7)–(10) whenever $x_1(0) \neq 0$.

Remark 8. The state x_1 satisfies the differential equation $\dot{x}_1 = -kx_1$; therefore, if $x_1(0) \neq 0$, we have $x_1(t) \neq 0$ for all $t \geqslant 0$. Thus, the surface $S_0 = \{x \in \mathbb{R}^3 \mid x_1 = 0\}$, where the control law is discontinuous, is never crossed by any trajectory of the closed-loop system, but is only asymptotically approached. This guarantees that the trajectories of the closed-loop system, starting out of S_0, are defined for all $t \geqslant 0$ and if $x_1(0) > 0$ ($x_1(0) < 0$) remain always in the semispace $S_+ = \{x \in \mathbb{R}^3 \mid x_1 > 0\}$ ($S_- = \{x \in \mathbb{R}^3 \mid x_1 < 0\}$).

Remark 9. The nonholonomic system described by Eq. (7) is of particular interest as such equations coincide locally (after state and input transformation) with the kinematic attitude equations of a rigid body free to rotate only around two axes [10] as well as with the kinematic equations describing the motion of a mobile robot [11].

Remark 10. Observe that, although system (7) is feedback equivalent to a three-dimensional chained system, to design a discontinuous control law yielding exponential convergence we do not need to transform it into chained form.

Fig. 1 shows the state trajectories and the control signals for the closed-loop system (7)–(10) with a prototype initial condition. Note the boundedness of the control signals and the exponential convergence of the state.

5.2. *Chained systems*

In this section we apply the proposed approach to design a discontinuous control law for an n-dimensional chained system, i.e. for a system described by equations of the form

$$\begin{aligned} \dot{x}_1 &= u_1, \\ \dot{x}_2 &= u_2, \\ \dot{x}_3 &= x_2 u_1, \\ \dot{x}_4 &= x_3 u_1, \\ &\vdots \\ \dot{x}_n &= x_{n-1} u_1. \end{aligned} \tag{11}$$

We deal with systems in chained form as they occupy a special place in the theory of nonholonomic control. In

[2] Note that the spectrum of the matrix A can be completely assigned through p_2 and p_3.

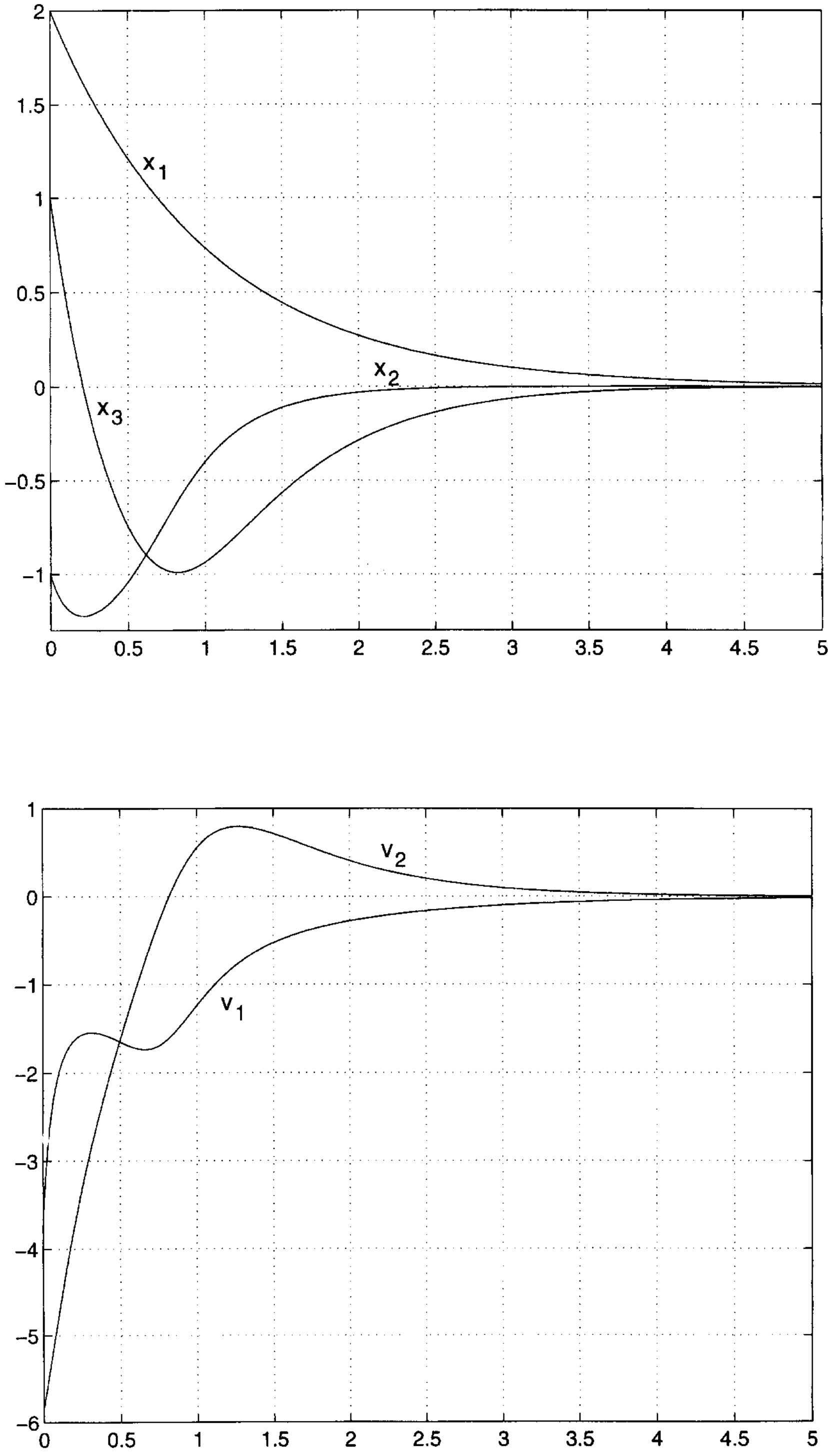

Fig. 1. State histories (top) and control signals (bottom) of a controlled knife edge from the initial condition [2, −1, 1].

fact, many nonholonomic mechanical systems (such as cars pulling trailers [18], nonholonomic manipulators [20], etc.) can be represented by kinematic models in chained form or are *feedback equivalent* to chained form.

Execute now the proposed algorithm.

(I) Apply the σ process

$$\begin{aligned} \xi_1 &= x_1, \\ \xi_2 &= x_2, \\ \xi_3 &= x_3/x_1, \\ &\vdots \\ \xi_{n-1} &= x_{n-1}/x_1^{n-3}, \\ \xi_n &= x_n/x_1^{n-2}. \end{aligned} \tag{12}$$

The transformed system is described by the equations

$$\begin{aligned} \dot{\xi}_1 &= u_1, \\ \dot{\xi}_2 &= u_2, \\ \dot{\xi}_3 &= \frac{\xi_2 - \xi_3}{\xi_1} u_1, \\ &\dots \\ \dot{\xi}_n &= \frac{\xi_{n-1} - (n-2)\xi_n}{\xi_1} u_1. \end{aligned} \tag{13}$$

(II) and (III) The linear control law

$$\begin{aligned} u_1 &= -k\xi_1, \\ u_2 &= p_2\xi_2 + p_3\xi_3 + \cdots + p_n\xi_n, \end{aligned}$$

with $k > 0$ and the coefficients p_i such that the matrix

$$A = \begin{bmatrix} p_2 & p_3 & p_4 & \dots & p_{n-1} & p_n \\ -k & k & 0 & \dots & 0 & 0 \\ 0 & -k & 2k & \dots & 0 & 0 \\ 0 & 0 & -k & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -k & (n-2)k \end{bmatrix} \tag{14}$$

has all eigenvalues with negative real part[3], globally exponentially stabilizes the discontinuous system (13).

(IV) In the x coordinates the control law is

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -kx_1 \\ p_2x_2 + p_3\dfrac{x_3}{x_1} + \cdots + p_n\dfrac{x_n}{x_1^{n-2}} \end{bmatrix}. \tag{15}$$

[3] Note that the spectrum of the matrix A can be completely assigned through the coefficients p_i's.

Remark 11. It is possible to show, see [3, Lemma 1], that the discontinuous control law (15) is well defined and bounded, for all $t \geqslant 0$, along the trajectories of the closed-loop system (11)–(15) whenever $x_1(0) \neq 0$.

Remark 12. As outlined in Remark 6 and as discussed in detail in [5], the discontinuous control law (15) is not a discontinuous exponential stabilizer for system (11). It only guarantees exponential convergence for all the initial conditions in the open and dense set $\{(x_1, \dots, x_n) \in \mathbb{R}^n \,|\, x_1 \neq 0\}$.

Remark 13. We stress again that asymptotic stability in the ξ coordinates does not imply asymptotic stability in the x coordinates as the σ process (12) does not map neighborhood of the origin of $\xi = 0$ into neighborhood of the origin of $x = 0$; see [5] for further detail.

Fig. 2 shows simulated results for the case $n = 4$. We set $k = 1$ and p_2, p_3 and p_4 such that the eigenvalues of the matrix (14) are all at $\lambda = -2$. Moreover we assume that $x_1(0) \neq 0$ and that the initial conditions $x_2(0)$ and $x_3(0)$ are small with respect to $x_1(0)$; this is without lack of generality, as discussed in [18].

6. Conclusions

In the present paper we have shown how the problem of asymptotic (exponential) convergence of nonholonomic systems can be easily solved, for a certain class of systems, using discontinuous control laws. The key idea is the use of a discontinuous coordinates transformation to *map* the initial system into the space of discontinuous nonholonomic systems. It turns out that in such a space the stabilization problem can be solved. The obtained control law is then transformed back to the starting coordinates system resulting in a discontinuous control. The theory has been applied to the stabilization of some prototype nonholonomic systems, for more examples, including use of σ processes different from those considered in this work, see [2,4].

The author thinks that the proposed approach can become helpful in solving a certain number of control problem, i.e. when theoretical results prevent the existence of smooth stabilizing control laws. Finally, we stress that one of the main and new ideas contained in the paper is the use of the σ process as a general *de-singularization* procedure.

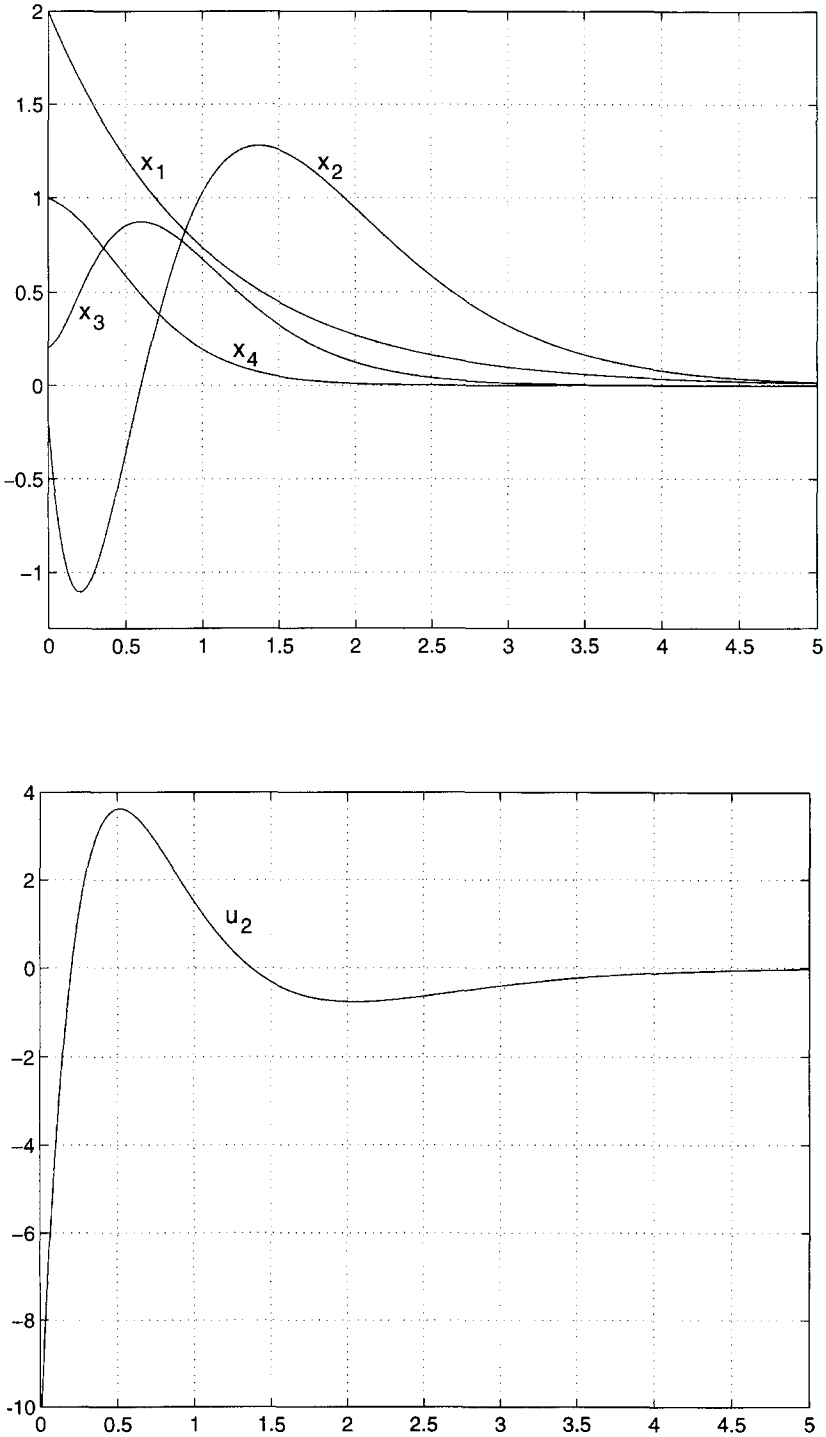

Fig. 2. State histories (top) and control signal u_2 (bottom) of a nonholonomic chained system of dimension four from the initial condition $[2, -0.2, 0.2, 1]$.

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