tions," *IEEE Trans. Automat. Contr.*, vol. AC-29, no. 9, pp. 831–832, Sept. 1984.

[17] C. A. Desoer and A. N. Gündes, "Bicoprime factorizations of the plant and their relations to right- and left-coprime factorizations," *IEEE Trans. Automat. Contr.*, vol. 33, no. 7, pp. 672–676, June 1988.

Exponential Stabilization of Mobile Robots with Nonholonomic Constraints

C. Canudas de Wit and O. J. Sørdalen

***Abstract*—This note presents an exponentially stable controller for a two-degree-of-freedom robot with nonholonomic constraints. Although this type of system is open-loop controllable, this system has been shown to be nonstabilizable via pure smooth feedback. In this note, a particular class of piecewise continuous controllers is shown to exponentially stabilize the mobile robot about the origin. This controller has the characteristic of not requiring infinite switching like other approaches, such as the sliding controller. Simulation results are also presented.**

I. Introduction

Path-tracking precision is essential for wheeled mobile robots (WMR) performing tasks such as welding, painting, gluing, drawing, etc., where some point in the WMR is in contact with the navigation surface and the WMR is to accurately follow a given path. Also good path-following capabilities are required when the robot has to move in environments with clusters of obstacles. The problem of controlling mobile robots with nonholonomic constraints has been addressed from two points of view.

1) Open-loop strategies that seek to find a bounded sequence of control inputs to steer the cart from any initial position to any other arbitrary configuration. The existence of such sequences has been indicated by [2]. Because these types of systems are locally controllable and reachable, [9] has proposed analytic tools based on Lie algebra and geometrical considerations to find the required control sequence. Reference [5] has worked on optimal sinusoid-type inputs for canonical systems, where controllability is obtained by first-order Lie brackets. Reference [11] has extended this work to noncanonical forms, requiring a high degree of bracketing to achieve controllability. This proposal results in suboptimal sinusoid-type inputs. These strategies have been studied in connection with the motion planning of mobile robots.

2) Closed-loop strategies consist of designing feedback loops stabilizing the cart about an arbitrary point in the state space. Although, the cart model is both locally controllable and reachable, it has been shown by [12] and [7], based on the work of [6] and [1], that there is no pure smooth state feedback law (i.e., C^∞) that can locally stabilize this class of systems. Nonlinear controllers for tracking a moving virtual cart (or reference cart) were proposed by [8] and [12], among others. The requirement of nonzero motion excludes the stabilization problem. Extension of the work of [12], including stopping phases, was studied by [13]. He proposed a continuous state feedback law that also depended on an exogeneous time variable. This control scheme yields asymptotic stabilization of the origin with a rate of $1/t$. An alternative to time-dependent smooth controllers is the discontinuous or piecewise smooth controllers. Reference [10] has proposed discontinuous controllers for Brockett's well-known example [6] and for the rigid spacecraft in failure mode. References [3] and [4] have presented a discontinuous controller for the knife-edge example. Their idea consists of first constructing an open-loop strategy to steer the system state from any initial condition to the origin. This results in a set of manifolds, which are then made invariant through a set of discontinuous feedbacks.

In this note we propose a "piecewise" smooth controller to render the origin exponentially stable for any initial condition in the state space. The main difference, with respect to other approaches, can be summarized as follows: the proposed scheme does not seek to render the discontinuous surface invariant, as opposed to the principles of sliding control, rather making this surface nonattractive. This consequently avoids infinite switching in the control law and the undesirable "chattering" phenomena. Furthermore, this control law yields exponential stability, and the convergence can be chosen arbitrarily fast.

II. Coordinate Transformation

The kinematics of a cart with two driving wheels are given as

$$\begin{aligned}\dot{x} &= \cos\theta(v_1+v_2)/2 = \cos\theta v\\ \dot{y} &= \sin\theta(v_1+v_2)/2 = \sin\theta v\\ \dot{\theta} &= (v_1-v_2)/(2c_r) = \omega\end{aligned} \tag{1}$$

where the state of the system (1) $q = [x, y, \theta]^T$ is the position of the wheel axis center (x, y) and the cart orientation θ with respect to the x-axis. The distance between the point (x, y) and each of the wheel locations is c_r. The velocities v_1 and v_2 are the tangent velocities of each wheel at its center of rotation (i.e., motor velocities times wheel radius). The control variables v and ω are, respectively, the tangent and angular cart velocities, and are related to the wheel velocities in the following manner:

$$u = \begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ \dfrac{1}{2c_r} & -\dfrac{1}{2c_r} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \tag{2}$$

The stabilization of system (1) is understood as designing a control law $u(q)$ so that the closed-loop system

$$\dot{q} = G(q)u(q) = f(q), \qquad G(q) = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \tag{3}$$

converges for any initial condition,[1] $q(0)$ to an equilibrium point in $\mathcal{O}$

$$\mathcal{O} = \{(x, y, \theta) = (0, 0, 2\pi n);\quad n = 0, \pm 1, \pm 2, \cdots\}.$$

[1] In general, the conditions for stabilization are not required to be global, but in this note we add this requirement.

Manuscript received May 17, 1991; revised October 18, 1991.

C. Canudas de Wit is with the Laboratoire d'Automatique de Grenoble, URA CNRS 228, ENSIEG—INPG, Saint-Martin-d'Hères, France.

O. J. Sørdalen is with the Laboratoire d'Automatique de Grenoble, URA CNRS 228, ENSIEG-INPG, Saint-Martin-d'Hères, France, on leave from the Division of Engineering Cybernetics, The Norwegian Institute of Technology, Trondheim, Norway.

IEEE Log Number 9203064.

Note that all points in $\mathcal{O}$ are equivalent in terms of positioning and orienting the cart.

Consider the following circle family $\mathcal{P}$:

$$\mathcal{P} = \{(x, y) : x^2 + (y - r)^2 = r^2\} \tag{4}$$

as the set of circles with radius $r = r(x, y)$. They pass through the origin and (x, y), and are centered on the y-axis with $\frac{\partial y}{\partial x} = 0$ in the origin. Let θ_d be the angle of the tangent of $\mathcal{P}$ at (x, y), defined as

$$\theta_d(x, y) = \begin{cases} 2\arctan(x/y) & ; \quad (x, y) \neq (0,0) \\ 0 & ; \quad (x, y) = (0,0). \end{cases} \tag{5}$$

θ_d is taken by convention to belong to $(-\pi, \pi]$. Hence, θ_d has discontinuities on the y-axis with respect to x. The discontinuity surface is defined as

$$\mathcal{D} = \{(x, y, \theta) : x = 0, y \neq 0\}. \tag{6}$$

In view of these definitions, we introduce the following change of coordinates:

$$a(x, y) = r\theta_d = \frac{x^2 + y^2}{y}\arctan(x/y) \tag{7}$$

$$\alpha(x, y, \theta) = e - 2\pi n(e), \qquad e = \theta - \theta_d(x, y) \tag{8}$$

where a is the arc length, and the orientation error $\alpha \in (-\pi, \pi]$ is a periodic and piecewise continuous function with respect to e. n takes values in $\{0, \pm 1, \pm 2, \cdots,\}$ in such a way that α belongs to $(-\pi, \pi]$. α is introduced so that all the elements in $\mathcal{O}$ are mapped into the unique point $(a, \alpha) = (0, 0)$. $\mathcal{E}$ is the set of the points in q where $\alpha(q)$ is discontinuous, i.e.,

$$\mathcal{E} = \{(x, y, \theta) : \alpha(x, y, \theta) = \pi\}. \tag{9}$$

Note that $a(x, y)$ defines the arc length from the origin to (x, y) along a circle centered on the y-axis and passes through these two points. $a(x, y)$ may be positive or negative, according to the sign of x. In the degenerate case, when $y = 0$, we define $a(x, 0) = 0$, which makes $a(x, y)$ continuous with respect to y. Discontinuities in $a(x, y)$ only take place on the y-axis. An illustration of these definitions is shown in Fig. 1.

Let us now introduce the function $F(\cdot)$: $R^3 \to R \times (-\pi, \pi]$, mapping the state-space coordinates $q \in R^3$ into the two-dimensional space $z \in R \times (-\pi, \pi]$

$$z = F(q); \qquad F(q) = \begin{bmatrix} a(x, y) \\ \alpha(x, y, \theta) \end{bmatrix}. \tag{10}$$

This transformation has several useful properties, which are listed in the following lemma.

Lemma 1: The mapping $F(\cdot): R^3 \to R \times (-\pi, \pi]$ has the following properties.

1) $F(0) = 0$.
2) $a^2(q)$, $\alpha^2(q)$, $\|F(q)\|^2$ are continuous in q.
3) $\|(x, y)^T\| \leq \|z\| \leq |\alpha| + |\alpha|$.

where $\|\cdot\|$ denotes the Euclidean norm.

Proof: The proof is simple and is left to the reader. □

Property 1 indicates the bijectivity between the equilibrium points in the q-space and the z-space. Property 2 indicates that the squares of the components of $F(\cdot)$ are continuous with respect to q. The last property results from the fact that the distance between two points on a circle is less than or equal to the arc length. These properties will later be useful in comparing the decreasing rates between these two quantities.

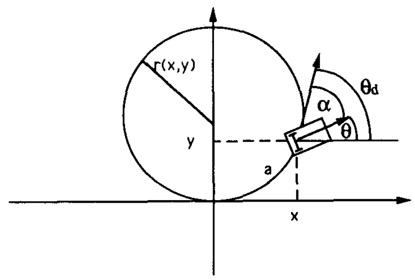

Fig. 1. Illustration of the coordinate transformation.

III. Control Design and Stability Analysis

This section proposes a piecewise smooth controller and analyzes the stability of the closed-loop system. First, the stability analysis is performed in an open continuous subspace, and is then extended to the whole state-space, including discontinuities.

A. Dynamics in the Ψ-Space

Let us first consider the case where $q \in \Psi$, where Ψ is defined as the open set $\Psi = R^3 - (\mathcal{D} \cup \mathcal{E})$. $F(\cdot)$ is differentiable in Ψ.

In Ψ we have

$$\dot{z} = \frac{\partial F}{\partial q}\dot{q} = J(q)\dot{q}; \qquad J(q) \in R^{2\times 3} \tag{11}$$

and $J(q)$, with $\beta = y/x$, is given as

$$J(q) = \begin{bmatrix} \frac{\theta_d}{\beta} - 1 & \frac{\theta_d}{2}\left(1 - \frac{1}{\beta^2}\right) + \frac{1}{\beta} & 0 \\ \frac{2\beta}{(1+\beta^2)x} & -\frac{2}{(1+\beta^2)x} & 1 \end{bmatrix} \tag{12}$$

together with (3), we get

$$\dot{z} = J(q)G(q)u = B(q)u; \qquad B = \begin{bmatrix} b_1 & 0 \\ b_2 & 1 \end{bmatrix} \tag{13}$$

with

$$b_1 = b_1(q) = \cos\theta\left(\frac{\theta_d}{\beta} - 1\right) + \sin\theta\left(\frac{\theta_d}{2}\left(1 - \frac{1}{\beta^2}\right) + \frac{1}{\beta}\right) \tag{14}$$

$$b_2 = b_2(q) = \cos\theta\frac{2\beta}{(1+\beta^2)x} - \sin\theta\frac{2}{(1+\beta^2)x}. \tag{15}$$

By noting that $\cos\theta = \cos(\alpha + \theta_d)$, $\sin\theta = \sin(\alpha + \theta_d)$; and $\cos\theta_d = (1 - \beta^2)/(1 + \beta^2)$, $\sin\theta_d = 2\beta/(1 + \beta^2)$, we can rewrite b_1 as

$$b_1(\alpha, \beta) = \cos\alpha + \left(-\sin\theta_d\left(\frac{\theta_d}{\beta} - 1\right) + \cos\theta_d\left(\frac{\theta_d}{2}\left(1 - \frac{1}{\beta^2}\right) + \frac{1}{\beta}\right)\right)\sin\alpha. \tag{16}$$

Lemma 2: The functions $b_i(q)$ have the following properties for any x and y, with $\beta = y/x$:

1)) $b_{\min}(\alpha) \le b_1(\alpha, \beta) \le b_{\max}(\alpha)$
2) $b_1(\alpha, \beta)$ is continuous in α
3) $\lim b_1(\alpha, \beta) = 1 \alpha \to 0$
4)$|b_2(q)a(q)| \le N$ for some constant $N > 0$

where

$$b_{\min}(\alpha) = \cos\alpha - \frac{\pi}{2}|\sin\alpha|$$

$$b_{\max}(\alpha) = \cos\alpha + \frac{\pi}{2}|\sin\alpha|.$$

Proof: See the Appendix. □

The properties in Lemma 2 will be useful in establishing the exponential stability of the closed-loop equations.

Taking the following control law, with $\gamma > 0$ and $k > 0$:

$$v = -\gamma b_1 a \tag{17}$$

$$\omega = -b_2 v - k\alpha \tag{18}$$

gives the following closed-loop equations:

$$\dot{a} = b_1 v = -\gamma b_1^2 a$$

$$\alpha = b_2 v + \omega = -k\alpha \tag{19}$$

which have the following solutions for $a(t)$ and $\alpha(t)$:

$$a(t) = a(0)\exp(-\gamma\kappa(t))$$

$$\alpha(t) = \alpha(0)\exp(-kt) \tag{20}$$

with

$$\kappa(t) = \int_0^t b_1^2(q(\tau))d\tau. \tag{21}$$

From these equations we have

$$\|z(t)\|^2 \le \|z(0)\|^2 \exp(-2\eta(t)) \tag{22}$$

where

$$\eta(t) = \min(\gamma\kappa(t), kt) \qquad \forall t \ge 0 \tag{23}$$

which indicates bounds on the norm of $z(t)$ in the continuous set Ψ.

The following subsection extends the boundness of $z(t)$ to the region including the discontinuities by showing that the discontinuous surfaces $\mathscr{D}$ and $\mathscr{E}$ are *not* invariant and that the norm of $z(t)$ remains constant when traversing the discontinuous surfaces. Explicit bounds on the exponential convergence rate of the norm of $z(t)$ are also given.

B. Dynamic Behavior on the Surfaces $\mathscr{D}$ and $\mathscr{E}$

Motion (or impossibility of motion) on the discontinuous surfaces $\mathscr{D}$ and $\mathscr{E}$ can be investigated by analyzing the direction of the vector field $f(q)$ from (3) in the neighborhood of the discontinuities.

Let us first consider the behavior on the surface $\mathscr{D}$.

Lemma 3: Any trajectory $q(t)$, solution of the closed-loop system

$$\dot{q} = f(q)$$

cannot stay in $\mathscr{D}$ in a closed time interval $I = [t_1, t_2]$, $t_2 > t_1$.

Proof: To prove that no motion is possible in $\mathscr{D}$, we can first compute $f(q)$ in the neighborhood of $\mathscr{D}$ as

$$f^+(q) = \lim_{x \to 0^+} f(q) = \begin{bmatrix} \cos\theta v^+ \\ \sin\theta v^+ \\ \omega^+ \end{bmatrix}$$

$$f^-(q) = \lim_{x \to 0^-} f(q) = \begin{bmatrix} \cos\theta v^- \\ \sin\theta v^- \\ \omega^- \end{bmatrix}$$

where

$$v^+ = -\gamma\left(-\cos\theta + \frac{\pi}{2}\sin\theta\right)\frac{\pi}{2}|y|$$

$$v^- = \gamma\left(-\cos\theta - \frac{\pi}{2}\sin\theta\right)\frac{\pi}{2}|y|$$

$$\omega^+ = -\cos\theta\gamma\pi\left(-\cos\theta + \frac{\pi}{2}\sin\theta\right)\mathrm{sgn}(y) - k(\theta - \pi - 2\pi n)$$

$$\omega^- = \cos\theta\gamma\pi\left(-\cos\theta - \frac{\pi}{2}\sin\theta\right)\mathrm{sgn}(y) - k(\theta - \pi - 2\pi n)$$

and then show that there is no convex combination of $f^-(q)$ and $f^+(q)$ that makes $q(t)$ stay in $\mathscr{D}$. In other words, there is not a $q \in \mathscr{D}$, $\delta \in [0, 1]$ and $\mu \in R$ such that

$$\mu f_i = \delta f^+(q) + (1 - \delta) f^-(q) \tag{24}$$

for all $t \in I$, where f_i indicates the directions of possible motions in $\mathscr{D}$. Note that in order to remain in $\mathscr{D}$ during a time interval I, the cart should either perform a motion along the y-axis, a pure rotation at a fixed point y, or stand still. By allowing μ to be equal to zero, which means that the cart stands still, these possibilities are represented by the following directions:

$$f_1 = [0, 1, 0]^T \qquad \text{or} \qquad f_2 = [0, 0, 1]^T.$$

The direction indicated by f_1 is equivalent to the situation where the cart is oriented in the y-direction, i.e., θ is constant and equal to 90° or 270°. In this case, it is simple to see that the last line in condition (24) cannot be satisfied. In the f_2-direction, where we have a pure rotation, the first two lines in condition (24) cannot be verified for all $t \in I$. □

We have shown that trajectories $q(t)$ cannot stay in $\mathscr{D}$. However, it should be noticed that $\mathscr{D}$ can be traversed. This is not the case for the surface $\mathscr{E}$, which is shown to be a repulsive discontinuity by the following lemma.

Lemma 4: Any trajectory $q(t)$, solution of the closed-loop system

$$\dot{q} = f(q)$$

starting in $\mathscr{E}$, i.e., $q(0) \in \mathscr{E}$, or in its neighborhood, will be repelled from $\mathscr{E}$.

Proof: To prove this, we only need to show that for any $q \in \mathscr{E}$, the projection of the vector field $f(q)$ on the normal of $\mathscr{E}$ points outwards from both sides of the surface. In other words, the inner products of $f(q)$ and the outpointing normal at each side of the discontinuous surface are strictly positive.

Let $s(q) = 0$ denote the discontinuity surface $\mathscr{E}$

$$s(q) = \theta - \theta_d(x, y) - 2\pi n - \pi.$$

Then the normal to $s(q) = 0$ is

$$n(q) = \frac{\partial s(q)}{\partial q} = \begin{bmatrix} \dfrac{2y}{x^2 + y^2} \\ -\dfrac{2x}{x^2 + y^2} \\ 1 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{r} \\ -\dfrac{x}{yr} \\ 1 \end{bmatrix}. \tag{25}$$

We define for $q \in \mathscr{E}$

$$f^+(q) = \lim_{s \to 0^+} f(q) = \begin{bmatrix} -\cos\theta_d \gamma r \theta_d \\ -\sin\theta_d \gamma r \theta_d \\ -\gamma\theta_d + k\pi \end{bmatrix}$$

$$f^-(q) = \lim_{s \to 0^-} f(q) = \begin{bmatrix} -\cos\theta_d \gamma r \theta_d \\ -\sin\theta_d \gamma r \theta_d \\ -\gamma\theta_d - k\pi \end{bmatrix}.$$

Then we have

$$\langle f^+, n \rangle = k\pi > 0 \tag{26}$$

$$\langle f^-, -n \rangle = k\pi > 0. \tag{27}$$

This means that in the neighborhood of $\mathscr{E}$, or when $q(0) \in \mathscr{E}$, the vector field $f(q)$ will always have a component driving the system away from $\mathscr{E}$. This can also be seen by studying the "potential" function, $V(s) = (1/2)s^2$.

$$\dot{V} = s\dot{s} = s < n, f^{\pm} > = s \cdot \operatorname{sgn}(s) 2\pi = 2\pi |s| \geq 0. \tag{28}$$

Therefore, $V(t)$ and, hence, $|s(t)|$ will always increase. □

C. Dynamics in the Complete Space

Lemmas 1, 3, and 4 allow us to extend the properties of the dynamics of the closed-loop system to the space including discontinuities. The following lemma summarizes these results.

Lemma 5: For any $q \in R^3$, $z \in R \times (-\pi, \pi]$, and $\forall t \geq 0$, we have

$$\|z(t)\| \leq \|z(0)\| e^{-\eta(t)} \tag{29}$$

$$\|a(t)\| \leq \|a(0)\| e^{-\kappa(t)} \tag{30}$$

$$\|\alpha(t)\| \leq \|\alpha(0)\| e^{-kt} \tag{31}$$

where $\kappa(t)$ and $\eta(t)$ are defined by (21) and (23).

The following theorem establishes our main result.

Theorem 1: There are positive constants, T, $\eta_0(T)$, $\sigma_0(T)$, and $\epsilon(T)$, so that the norm of $z(t)$ satisfies

$$\|z(t)\|^2 \leq \sigma_0^2(T) e^{-2\eta_0(T)t}, \qquad \forall t \geq 0 \tag{32}$$

where $1 > \epsilon(T) > 0$; and $\sigma_0(T)$, $\eta_0(T)$ are given as

$$\eta_0(T) = \min(\gamma(1 - \epsilon(T)), k) \tag{33}$$

$$\sigma_0(T) = \max(|a(0)| e^{\gamma(1-\epsilon(T))T}, |\alpha(0)|) \tag{34}$$

with arbitrary, positive constants γ and k.

Proof: Lemma 2 gives upper and lower bounds on $b_1(\alpha, \beta)$ and shows that when α approaches zero, $b_1(\alpha, \beta)$ continuously tends towards one. Lemma 5 shows that $\alpha(t)$ decreases exponentially to zero. Therefore, for all $t \geq T$, there is a small enough $\epsilon(T)$ so that

$$|b_1^2(\alpha(t), \beta(t)) - 1| \leq \epsilon(T), \qquad \forall t \geq T$$

which gives the following bounds on b_1^2:

$$1 - \epsilon(T) \leq b_1^2(\alpha(t), \beta(t)) \leq 1 + \epsilon(T), \qquad \forall t \geq T.$$

In view of Lemma 5, we have for all $t \geq 0$

$$\begin{aligned} |a(t)| &\leq |a(0)| e^{-\kappa(t)} \\ &\leq |a(0)| e^{-\gamma\int_T^t (1-\epsilon(T))d\tau - \gamma\int_0^T b_1^2(\tau)d\tau} \\ &\leq |a(0)| e^{-\gamma(1-\epsilon(T))(t-T)} \\ &= |a(0)| e^{\gamma(1-\epsilon(T))T} e^{-\gamma(1-\epsilon(T))t} \\ &= a_0(T) e^{-\gamma(1-\epsilon(T))t} \end{aligned}$$

where

$$a_0(T) = |a(0)| e^{\gamma(1-\epsilon(T))T}.$$

And therefore

$$\begin{aligned} \|z(t)\|^2 &= a^2(t) + \alpha^2(t) \\ &\leq a_0^2(T) e^{-2\gamma(1-\epsilon(T))t} + \alpha^2(0) e^{-2kt} \\ &\leq \max(a_0^2(T), \alpha^2(0)) e^{-2\min(\gamma(1-\epsilon(T))), kt} \\ &\leq = \sigma_0^2(T) e^{-2\eta_0(T)t}. \end{aligned}$$

□

It can now be established that exponential convergence of $z(t)$ to zero implies exponential convergence of the q-trajectories to any of the members of $\mathscr{O}$.

Theorem 2: For any initial condition $q(0) \in R^3$, the solution $q(t)$, $t > 0$, of the closed-loop system

$$\dot{q} = f(q)$$

exponentially converges to any of the elements in $\mathscr{O} = \{(0, 0, 2\pi n), \quad n = 0, \pm 1, \pm 2, \cdots, \}$.

Proof: The proof will be based on the basic properties of the norms of q and z. Note first, from Lemma 1, property 3, that the distance from (x, y) to the origin is upper bounded by the arc length $|a|$

$$\|(x(t), y(t))^T\|^2 \leq |a(t)|^2, \qquad \forall t \geq 0.$$

Since $a(t)$ tends exponentially towards zero, the norm of $[x, y]^T$ will converge exponentially to zero. It remains to show that the cart orientation θ converges to a point in $\mathscr{O}$. For this purpose, we recall that θ can be written as a function of α as (8)

$$\theta(t) = \alpha(t) + \theta_d(t) + 2\pi n$$

where the variable n increments when the y-axis (or $\mathscr{D}$) is crossed from the right to the left, and decrements when $\mathscr{D}$ is traversed in the opposite direction.

Since $\alpha(t)$ tends exponentially to zero, the behavior of $\theta(t)$ will be determined by the behavior of $\theta_d(t)$. θ_d is by definition the tangent angle to the circles defined by $\mathscr{P}$, and α is the error between the actual orientation and this tangent angle. Since $\alpha(t)$ converges exponentially to zero, the motion of the cart will exponentially converge to a motion along one of the circles defined by $\mathscr{P}$. Theorem 1 showed that the distance $|a|$ from the origin to the position of the cart along this circle, converges exponentially to zero. We, therefore, have the position of the cart exponentially tending towards the origin along a circle. Therefore, $\theta_d(x(t), y(t))$ exponentially converges to its limit $\theta_d(0, 0) = 0$. Since $\theta(t)$ converges exponentially to θ_d, $\theta(t)$ will also converge exponentially to zero. □

Corollary 1: The control inputs $v(q)$ and $\omega(q)$ remain bounded for any $q \in R^3$.

Proof: Boundness of v and ω follows from the properties 1) and 4) of b_1 and $b_2 a$ listed in Lemma 2, and the fact that a and α are bounded quantities. It should also be observed that both the inputs v and ω tend towards zero as time goes to infinity. □

Theorem 1 gives bounds on the convergence rate η_0 and on the magnitude of the norm of $z(t)$, σ_0. Note, however, that when T is high, σ_0 may describe a too-conservative bound on $\|z(t)\|$ since σ_0 grows exponentially as T increases. Design guidelines for choosing the control gains can be established from Lemma 5 and Theorem 1.

D. Design Guidelines

A suitable design specification may be to give time intervals T_a and T_α as the time needed to decrease $a(t)$ and $\alpha(t)$, respectively, from their initial values $a(0)$ and $\alpha(0)$, to some specified values $a(T_a)$ and $\alpha(T_\alpha)$, relative to $a(0)$ and $\alpha(0)$, i.e.,

$$\left|\frac{\alpha(T_\alpha)}{\alpha(0)}\right| \le n_\alpha, \qquad \forall t \ge T_\alpha$$

and

$$\left|\frac{a(T_a)}{a(0)}\right| \le n_a, \qquad \forall t \ge T_a > T_\alpha$$

where n_α and n_a describe these desired rates. From these quantities it is straightforward to compute the corresponding value of k as

$$k = -\frac{1}{T_\alpha}\ln(n_\alpha).$$

From T_α on, when α is small, bounds on $b_1^2(\alpha, \beta)$ can be approximated as (the first property in Lemma 2)

$$1 - \left(-\alpha^2\left(\frac{\pi^2}{4} - 1\right) + \pi|\alpha|\right) \le b_1^2(\alpha, \beta) \le 1 + \left(\alpha^2\left(\frac{\pi^2}{4} - 1\right) + \pi|\alpha|\right). \quad (35)$$

Hence, $\forall t \ge T_\alpha$, $\epsilon(T_\alpha)$ can be taken as

$$\epsilon(T_\alpha) = \alpha^2(T_\alpha)\left(\frac{\pi^2}{4} - 1\right) + \pi|\alpha(T_\alpha)|$$

$$\epsilon(T_a) \le n_\alpha^2\pi^2\left(\frac{\pi^2}{4} - 1\right) + \pi^2 n_\alpha = \epsilon_0 < 1$$

where we have used that $|\alpha(0)| \le \pi$. ϵ_0 can be chosen arbitrarily small by choosing n_α small enough. From Lemma 5, we can thus compute a value for γ as

$$|a(T_a)| \le |a(0)|e^{-\gamma(1-\epsilon(T_\alpha))(T_a - T_\alpha)}$$

$$|a(T_a)| \le |a(0)|e^{-\gamma(1-\epsilon_0)(T_a - T_\alpha)}.$$

Hence

$$\gamma = -\frac{1}{(1-\epsilon_0)(T_a - T_\alpha)}\ln(n_a). \quad (36)$$

Both the constants k and γ are thus functions of T_a, T_α, n_a, and n_α. For example, $T_a = 5T_\alpha = 5$, $n_\alpha = n_a = 0.01$ give

$$k = 4.60, \qquad \gamma = 1.35.$$

E. Simulations

Simulations were done by using the SIMNON package [17]. The constant k and γ were chosen to 4.60 and 1.35, respectively. Fig. 2 shows the resulting paths in the xy-phase plane for several initial conditions corresponding to different points on the unit circle with an initial orientation angle $\theta(0) = \pi/2$. We see that all these phase trajectories are asymptotically stable and reach the origin by asymptotically tracking one of the circles in the path family $\mathscr{P}$.

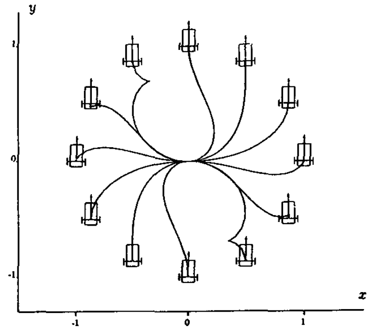

Fig. 2. Resulting paths when the cart is initially on the unit circle in the xy-plane with $\theta(0) = \pi/2$.

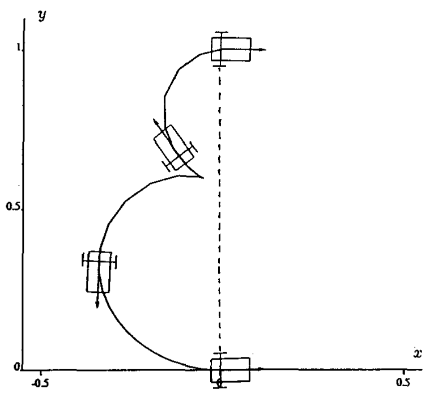

Fig. 3. The resulting path in the xy-plane with $q(0) = [0, 1, 0]^T$.

Fig. 3 shows the closed-loop resulting path for $q(0) = [0, 1, 0]^T$. Note that the cart starts at the discontinuity surface $\mathscr{D}$ and asymptotically converges to the origin. This is a kind of parking maneuver. The shape of the path will vary with variations in the control gains k and γ. Fig. 4 shows another example where the cart starts at the left of the surface $\mathscr{D}$, $q(0) = [-0.05, 1, \pi/2]^T$, and crosses it only once before reaching the origin. Fig. 5 shows the corresponding time histories of $x(t)$, $y(t)$, and $\theta(t)$, illustrating asymptotic convergence to 0. Fig. 6 illustrates the discontinuity in the inputs v and ω, when the y-axis is crossed. We note that the inputs are bounded, and chattering is avoided.

IV. Conclusions

A piecewise smooth controller has been proposed for a mobile robot with two degrees of freedom. The particularity of this controller is that infinite high-frequency components and the well-known problem of "chattering" are avoided. The cart exponentially converges to the origin with zero orientation for any initial condition. This is achieved by letting the motion of the cart converge to one of the circles, which pass through the origin and are centered on the y-axis. The circles were chosen because they yield a new change of coordinates which is geometrically meaningful. However, other types of paths may also be possible.

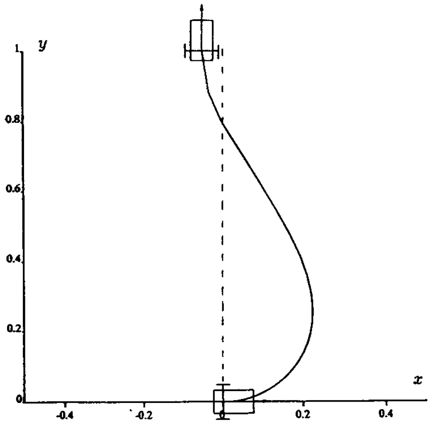

Fig. 4. A path crossing the line of discontinuity, the y-axis. $q(0) = [-0.05, 1, \pi/2]^T$.

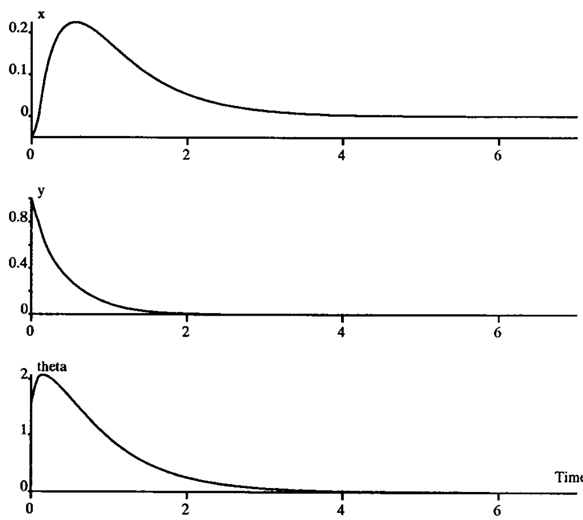

Fig. 5. Timeplots of the position $x(t)$ and $y(t)$ and the orientation $\theta(t)$ for a path crossing the y-axis, see Fig. 4.

Common for these paths is that they must pass through the origin with the desired derivative so that the desired orientation is asymptotically reached.

Appendix
Proof of Lemma 2

1) From (16) we have, with $\beta = y/x$

$$b_1(\alpha, \beta) = \cos\alpha + B(\beta)\sin\alpha \tag{37}$$

where

$$B(\beta) = -\sin\theta_d\left(\frac{\theta_d}{\beta} - 1\right) + \cos\theta_d\left(\frac{\theta_d}{2}\left(1 - \frac{1}{\beta^2}\right) + \frac{1}{\beta}\right)$$

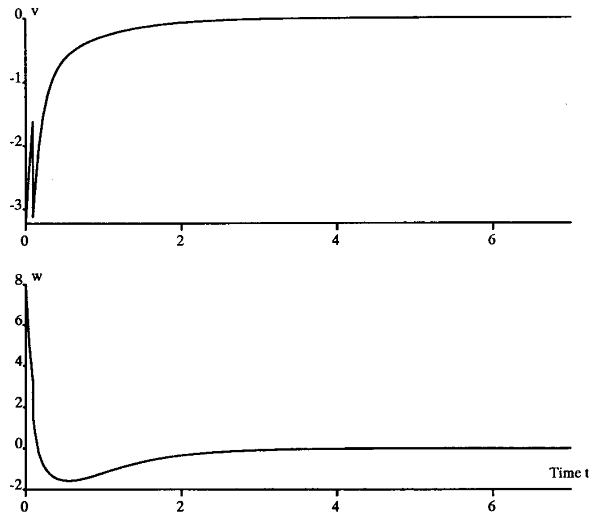

Fig. 6. Timeplots of the inputs, $v(t)$ (tangential velocity) and $\omega(t)$ (rotational velocity), for a path crossing the y-axis; see Fig. 4. $v(t)$ and $\omega(t)$ become discontinuous when $x = 0$, $y \neq 0$.

$$= -\sin\theta_d\left(\frac{\theta_d}{\beta} - 1\right) + \cos\theta_d\left(\theta_d\frac{\beta^2 - 1}{2\beta} + 1\right)\frac{1}{\beta}$$

$$= -\sin\theta_d\left(\frac{\theta_d}{\beta} - 1\right) + \cos\theta_d\left(1 - \frac{\theta_d}{\tan\theta_d}\right)\frac{1}{\beta}$$

$$= -\sin\theta_d\left(\frac{\theta_d}{\beta} - 1\right) + \cos\theta_d\frac{1}{\beta} - \frac{\cos^2\theta_d}{\sin\theta_d}\frac{\theta_d}{\beta}$$

$$= -\sin\theta_d\left(\frac{\theta_d}{\beta} - 1\right) + \cos\theta_d\frac{1}{\beta} + \sin\theta_d\frac{\theta_d}{\beta} - \frac{1}{\sin\theta_d}\frac{\theta_d}{\beta}$$

$$= \sin\theta_d + \cos\theta_d\frac{1}{\beta} - \frac{1}{\sin\theta_d}\frac{\theta_d}{\beta}$$

$$= \frac{2\beta}{1+\beta} + \frac{1-\beta^2}{1+\beta^2}\frac{1}{\beta} - \frac{1+\beta^2}{2\beta}\frac{2\arctan\beta}{\beta}$$

$$= \frac{1}{\beta} - \left(1 + \frac{1}{\beta^2}\right)\arctan\beta$$

$$\delta_d = \theta_d(\beta)\text{-}2\arctan(\beta).$$

Here, we have used the fact that

$$\cos\theta_d = \frac{1-\beta^2}{1+\beta^2}, \qquad \sin\theta_d = \frac{2\beta}{1+\beta^2}, \qquad \tan\theta_d = \frac{2\beta}{1-\beta^2}.$$

In order to find the maximum and minimum values of $B(\beta)$, we analyze the derivative, $B'(\beta)$

$$B'(\beta) = -\frac{1}{\beta^2} + \frac{2}{\beta^3}\arctan\beta - \left(1 + \frac{1}{\beta^2}\right)\frac{1}{1+\beta^2}$$

$$B'(\beta) = \frac{2}{\beta^2}\left(\frac{\arctan\beta}{\beta} - 1\right) < 0 \qquad \forall\beta.$$

We note that $B'(\beta)$ is continuous in $\beta = 0$, if we define $B'(0) = -(2/3)$.

Since $B'(\beta)$ is negative for all $\beta \in R$, we find that

$$\lim_{\beta \to \infty} B(\beta) \le B(\beta) \le \lim_{\beta \to -\infty} B(\beta)$$
$$\Downarrow$$
$$-\frac{\pi}{2} \le B(\beta) \le \frac{\pi}{2}.$$

Therefore, we get

$$\cos\alpha - \frac{\pi}{2}|\sin\alpha| \le b_1(\alpha, \beta) \le \cos\alpha + \frac{\pi}{2}|\sin\alpha|.$$

2) From 1) we have that

$$b_1(\alpha, \beta) = \cos\alpha + B(\beta)\sin\alpha$$

where $B(\beta)$ is bounded. Since $\cos\alpha$ and $\sin\alpha$ are continuous in α, and $B(\beta)$ is bounded, it is clear that $b_1(\alpha, \beta)$ is also continuous in α.

3) By the continuity property from 2) we find.

$$\lim_{\alpha \to 0} b_1(\alpha, \beta) = \lim_{\alpha \to 0} \cos\alpha + \lim_{\alpha \to 0} B(\beta)\sin\alpha = 1.$$

4) From the definition of $a(x, y)$ (7) we have

$$a = r\theta_d = \frac{x^2 + y^2}{y}\arctan\frac{y}{x} = x\frac{1 + \beta^2}{\beta}\arctan\beta.$$

From the definition of b_2 (15) we get

$$|b_2 a| = |\left(\cos\theta\frac{2\beta}{(1 + \beta^2)x}\rho - \sin\theta\frac{2}{(1 + \beta^2)x}\right) \cdot x\frac{1 + \beta^2}{\beta}\arctan\beta|$$

$$|b_2 a| = |2\cos\theta\arctan\beta - 2\sin\theta\frac{\arctan\beta}{\beta}|$$

$$|b_2 a| \le \pi|\cos\theta| + 2|\sin\theta| \le \pi + 2 = N. \qquad \square$$

References

[1] D. Aeyels, "Stabilization of a class on nonlinear systems by smooth feedback," in *Syst. Contr. Lett.*, no. 5, pp. 289–294, 1985.

[2] J. Barraquand and J.-C. Latombe, "On nonholonomic mobile robots and optimal maneuvering," presented at the *4th Int. Conf. Intelligent Cont. Conf.*, Albany, NY, 1989.

[3] A. M. Bloch and N. H. McClamroch, "Control of mechanical systems with classical nonholonomic constraints," presented at the *Conf. Decision Cont.*, Tampa, FL, Dec. 1989.

[4] A. M. Bloch, N. H. McClamroch, and M. Reyhanoglu, "Controllability and stabilizability properties of a nonholonomic control systems," presented at the *Conf. Decision Contr.*, Honolulu, HI, Dec. 1990.

[5] R. W. Brockett, "Control theory and singular Riemannian geometry," in *New Directions in Applied Mathematics*. New York: Springer-Verlag, 1981, pp. 11–27.

[6] ——, "Asymptotic stability and feedback stabilization," in *Differential Geometric Control Theory*. R. W. Brockett, R. S. Millman, and H. J. Sussman, Eds. Boston, MA: Birkhauser, 1983, pp. 181–208.

[7] G. Campion, B. d'Andrea-Novel, and G. Bastin, "Controllability and state feedback stabilizability of nonholonomic mechanical systems," in *Advanced Robot Control*; *Proc. Int. Workshop on Nonlinear and Adaptive Contr.: Issues in Robotics*. Grenoble, Nov. 21–23, 1990, vol. 162, C. Canudas de Wit, Ed. New York: Springer-Verlag, 1991.

[8] V. Kanayama, V. Kimura, F. Miyazaki, and T. Noguchi, "A stable tracking control method for an autonomous mobile robot," *IEEE Conf. Robotics Automat.*, Cincinnati, OH, 1990, pp. 384–389.

[9] G. Lafferriere and H. J. Sussmann, "Motion planning for controllable systems without drift," in *Proc. 1991 IEEE Int. Conf. Robotics Automat.*, Sacramento, CA, Apr. 1991, pp. 1148–1153.

[10] F. Messager, "Two nonlinear examples of discontinuous stabilization," in *Colloque International Sur L'analyse des Systemes Dynamiques Controles*, vol. 1, Lyon, France, July 1990.

[11] R. M. Murray and S. S. Sastry, "Steering Nonholonomic Systems Using Sinusoids," presented at the *IEEE Conf. Decision Contr.*, Honolulu, HI, Dec. 1990.

[12] C. Samson and K. Ait-Abderrahim, "Feedback control of a nonholonomic wheeled cart in Cartesian space," in *Proc. of the 1991 IEEE, Int. Conf. Robotics Automat.*, Sacramento, CA, Apr. 1991.

[13] C. Samson, "Velocity and torque feedback control of a nonholonomic cart," in *Advanced Robot Control*; *Proc. Int. Workshop on Nonlinear and Adaptive Contr.: Issues in Robotics*, Grenoble, Nov. 21–23, 1990, vol. 162, C. Canudas de Wit, Ed. New York: Springer-Verlag, 1991.

[14] H. Elmqvist, K. J. Åström, T. Schönthal, and B. Wittenmark, "SIMNON user's guide for MS-DOS computers," version 3.0, SSPA Systems, Göteborg, Sweden, Jan. 1990.

A Simplified Approach to Bode's Theorem for Continuous-Time and Discrete-Time Systems

Bing-Fei Wu and Edmond A. Jonckheere

Abstract—**This note presents a simplified approach to Bode's theorem for both continuous-time and discrete-time systems, along with some generalization. For continuous-time systems, the constraints of open-loop stability and roll-off at $s = \infty$ are removed. A counterexample shows that, when the excess poles / zeros vanishes, the Bode integral drops from infinite to finite value when the open-loop gain crosses a critical value. A revised result is also developed in this note. The salient feature of this approach is that at no stage do we invoke either Cauchy's theorem or the Poisson integral; the simplified proof relies only on elementary analysis. This approach carries over to the discrete-time case in a straightforward manner.**

I. Introduction

Bode's theorem [2, p. 285] states that for an open-look stable transfer function with the difference between degrees of numerator and denominator at least 2, the integral over all frequencies of the natural log of the magnitude of the sensitivity function $\ln|S(j\omega)|$ vanishes. This result reveals that it is not, in general, possible to decrease $|S(j\omega)|$ below the threshold value of 1 over all frequencies. Freudenberg and Looze [4] extended Bode's theorem to unstable open-loop systems. They showed that the integral of the log of the magnitude of the sensitivity function is proportional to the sum of the unstable open-loop poles. Kwakernaak and Sivan [8, pp. 440–441] pointed out that if the open-loop system is asymptotically stable, then this integral could be zero, finite, or infinite, depending on the degree difference between numerator and denominator of the open-loop transfer function.

The main result of this note is a simplified derivation of two new Bode's theorems. One is called *revised generalized Bode's*

Manuscript received May 17, 1991; revised September 6, 1991. This work was supported by The Air Force Office of Scientific Research under Grant 89-0398.

The authors are with the Department of Electrical Engineering–Systems, University of Southern California, Los Angeles, CA 90089.

IEEE Log Number 9203065.