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Exponential stabilization of nonholonomic mobile robots

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Abstract

The problem of point-to-point control design for differentially steered nonholonomic mobile robots is considered in this paper. The control variables are derived using Lyapunov's stability technique and are piecewise continuous. The proposed control law guarantees the exponential stability of the closed-loop system and ensures the convergence of the position and the orientation of the robot to their desired fixed values. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Mobile robot; Nonholonomic constraint; Point-to-point motion; Stabilization and control; Piecewise continuous control; Backstepping technique; Lyapunov stability method; Exponential stability

1. Introduction

Nonholonomic systems are a class of mechanical systems with nonintegrable constraints [1]. Examples of such systems include mobile robots, multi-cart trailers, under-actuated mechanical manipulators, under-actuated spacecrafts, autonomous surface vessels and rockets [1–6]. Due to Brockett's condition [7], nonholonomic systems cannot be asymptotically stabilized using continuous static state feedback controls [1,7]. There are many reports in the literature addressing the stabilization problem for nonholonomic systems [3,8–10]. Most of these reports, in accordance with Brockett's condition, employ time varying, or discontinuous, or dynamic state feedback controller designs. In Ref. [3], Canudas de Wit et al. provided several solutions for the stabilization problem, including time-varying control, piecewise continuous control and time-varying piecewise continuous control. Jiang and Pomet [8] introduced a class of time-varying smooth stabilizing feedback control schemes, resulting in a globally marginally stable closed-loop system. Guldner and Utkin [9] used sliding mode control techniques, resulting in discontinuous controls

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that guaranteed exponential stability of the closed-loop system. In Ref. [10], Mukherjee et al. introduced dynamic state feedback control that guaranteed stabilization to a given posture.

Taking a different approach, in Refs. [11–13] it is shown that, using a discontinuous transformation, a class of piecewise continuous control schemes can be derived for the stabilization of a nonholonomic mobile robot to a fixed posture. An example of such a transformation is the polar coordinates, in which the stabilization problem can be solved much easier and more intuitively. The resulting control is generally continuous at all points, except on a discontinuity surface that the robot may cross a finite number of times.

In this paper, motivated by the approach used in Refs. [11–13], the kinematic model of the robot is first transformed into a feasible coordinate system. A new control scheme is then derived for the stabilization problem in the new coordinate system. The controls are piecewise continuous and guarantee the exponential stability of the closed-loop system.

2. Kinematic model of the mobile robot

The kinematic model of a mobile robot with two differentially driven rear wheels and a castor front wheel is given by the drift-free equation. (1)

$$\dot{q} = f(q,u) = G(q)u \tag{1}$$

where $q = [x, y, \phi]^{\mathrm{T}}$ is the state vector, $u = [v, \omega]^{\mathrm{T}}$ is the input vector, and that

$$G(q) = \begin{bmatrix} \cos\phi & 0 \\ \sin\phi & 0 \\ 0 & 1 \end{bmatrix}.$$

Equivalently, this can be written as

$$\dot{x} = v\cos\phi \tag{2}$$

$$\dot{y} = v\sin\phi \tag{3}$$

$$\dot{\phi} = \omega \tag{4}$$

Here, the state vector $q = [x, y, \phi]^{\mathrm{T}}$ denotes the generalized position (position and orientation) of the robot with respect to a fixed reference frame and the control vector $u = [v, \omega]^{\mathrm{T}}$ denotes the translational and rotational velocities of the robot. It is assumed that the wheels of the robot do not slide. This is expressed by the nonholonomic constraint

$$\dot{x}\sin\phi - \dot{y}\cos\phi = 0 \tag{5}$$

3. Statement of the problem

Consider a wheeled mobile robot with the kinematic model given in Eq. (1) where its center of mass is positioned at point $R = [x, y]^{\mathrm{T}}$ and that its orientation angle is ϕ. It is desired to design a stable point-to-point control algorithm to drive the robot from any arbitrary point to another.

Without any loss of generality, one can always consider the target point to be the origin. Here, the objective is to find a piecewise continuous control vector $u = [v, \omega]^T$ so that the robot's position $R = [x, y]^T$ and orientation ϕ exponentially approach the desired target position $R_d = [x_d, y_d]^T = 0$ and the orientation angle of $\phi_d = 0$. That is, the designed controller is to drive the state vector $q = [x, y, \phi]^T$ of the robot to the desired position $q_d = [x_d, y_d, \phi_d]^T = [0, 0, 0]^T$, exponentially. In the sequel, without any loss of generality, the generalized position vector, q, is also considered as the position error.

4. Stabilizing controller design

Here, a stabilizing controller is designed which can drive a wheeled mobile robot with an arbitrary generalized position $q = [x, y, \phi]^T$ to exponentially arrive at the desired position $q_d = [x_d, y_d, \phi_d]^T = [0, 0, 0]^T$, that is at the origin with the heading angle of zero, as shown in Fig. 1. The technique used here is to force the heading angle (direction) ϕ of the robot approach an auxiliary direction ϕ_a, which points at an auxiliary target point M on the x-axis and on the same side of the plane as the robot R. The auxiliary direction ϕ_a, on the other hand, approaches the desired orientation $\phi_d = 0$, as the robot gets closer to the target. The robot R therefore must approach the auxiliary target M and, hence, the desired target R_d along a trajectory where its orientation approaches the desired value $\phi_d = 0$. To achieve this, we define a transformation that converts the robot's kinematic Eqs. (2)–(4) to a form that is suitable for our control development.

From Fig. 1, it is easy to find that

$$d = \sqrt{x^2 + y^2} \tag{6}$$

$$\xi = \phi_a - \phi \tag{7}$$

$$\eta = s_x \tan^{-1}\left(\frac{y}{|x|}\right) \tag{8}$$

where $\phi_a = (2/b)\eta$, $1 < b < 2$, $-\pi/2 \leqslant \eta \leqslant \pi/2$, and that by definition

$$s_x = \mathrm{sign}(x) = \begin{cases} 1, & x \geqslant 0 \\ -1, & x < 0 \end{cases}.$$

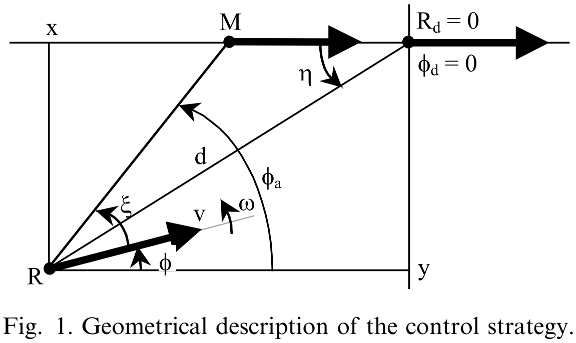

Fig. 1. Geometrical description of the control strategy.

Note that, for any point $R = [x, y]^T$ in the plane there exists a unique vector $[d, \eta]^T$. Let us now define $p = [d, \xi, \phi]^T$ as an equivalent state vector for the robot. Since ϕ is an independent variable, for any state vector $q = [x, y, \phi]^T$ there must exist a unique vector $p = [d, \xi, \phi]^T$. That is, any generalized position vector $q = [x, y, \phi]^T$ is uniquely mapped into the equivalent vector $p = [d, \xi, \phi]^T$. Since $p = 0$ implies $q = 0$, in our controller design we can indirectly achieve $q = 0$ by forcing $p = 0$. Using the above transformation, the kinematic model of the mobile robot can be rewritten as

$$\dot{d} = s_x \psi_1 v \tag{9}$$

$$\dot{\xi} = -s_x \frac{2}{b} \psi_2 \frac{v}{d} - \omega \tag{10}$$

$$\dot{\phi} = \omega \tag{11}$$

where $\psi_1 = \cos(\eta - \phi)$ and $\psi_2 = \sin(\eta - \phi)$. Now note that, η has discontinuities on the y-axis with respect to x. In addition, ϕ and $\xi_1 = a\xi$ are constrained to have equivalent modulus values from $(-\pi, \pi]$, where $a = 1 + 2((b-1)/\pi)|\eta|$. The discontinuity surfaces for η, ϕ and ξ are expressed as

$$D = D_1 \cup D_2 \cup D_3 \tag{12}$$

where $D_1 = \{p\colon b(\xi + \phi) = \pm\pi\}$, $D_2 = \{p\colon \phi = \pm\pi\}$, $D_3 = \{p\colon a\xi = \pm\pi\}$ and that $p = [d, \xi, \phi]^T$. Now consider the control algorithm, given by

$$v = -c_1 s_x \frac{\psi_1}{\psi_1^2 + \xi^2} d \tag{13}$$

$$\omega = c_2 \xi + \frac{c_1}{\psi_1^2 + \xi^2} \left(\frac{2}{b} \psi_1 \psi_2 + \xi d^2 \right) \tag{14}$$

where c_1 and c_2 are arbitrary positive constants. The control vector $u = [v, \omega]^T$ is continuous everywhere except at a finite number of points where the robot may cross the discontinuity surface D. However, the discontinuity surface D is not an invariant set, and hence, the robot cannot get stuck at points in D. Moreover, the control signals, v and ω, are bounded everywhere. The proposed controller results in a closed-loop system as

$$\dot{d} = -c_1 \frac{\psi_1^2}{\psi_1^2 + \xi^2} d \tag{15}$$

$$\dot{\xi} = -\left(c_2 + c_1 \frac{d^2}{\psi_1^2 + \xi^2} \right) \xi \tag{16}$$

$$\dot{\phi} = c_2 \xi + \frac{c_1}{\psi_1^2 + \xi^2} \left(\frac{2}{b} \psi_1 \psi_2 + d^2 \xi \right) \tag{17}$$

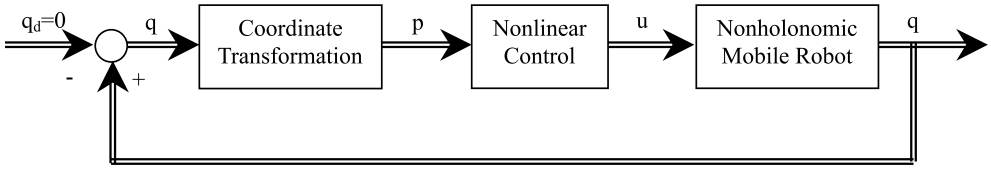

Fig. 2. Block diagram of the proposed controller.

The block diagram of the closed-loop system with the proposed controller is shown in Fig. 2.

It can be shown that the above closed-loop system is globally exponentially stable. These are summarized in the following result.

Result. *Consider a mobile robot defined by the kinematic model given in Eq*. (5). *Then the closed-loop system with control* $u = [v, \omega]^{\mathrm{T}}$, *given in Eqs*. (13) *and* (14), *is globally exponentially stable. Moreover*, *the control vector is piecewise continuous and bounded.*

Proof. Consider a Lyapunov function candidate given by

$$V(d, \xi) = \tfrac{1}{2}d^2 + \tfrac{1}{2}\xi^2 \tag{18}$$

which is a positive definite function and is radially unbounded. The time derivative of this function along the trajectories of the system is given by

$$\dot{V} = d[s_x\psi_1 v] + \xi\left[-s_x\frac{2}{b}\psi_2\frac{v}{d} - \omega\right] \tag{19}$$

Substituting for v and ω into $\dot{V}$, using Eqs. (13) and (14), we get

$$\dot{V} = -c_1 d^2 - c_2\xi^2 \tag{20}$$

which is negative definite. This implies that V, and hence, d and ξ are bounded everywhere [14,15]. However, since ξ is discontinuous, this implies that both d and ξ exponentially converge to the largest invariant set in $E \cup \{0\}$, where $E = \{[d, \xi]^{\mathrm{T}}\colon p \in D\} \subseteq D$. Also, since $1 < b < 2$ and $-\pi/2 \leqslant \eta \leqslant \pi/2$, ξ and ψ_1 can never be zero simultaneously, and hence, $\psi_1^2 + \xi^2 \neq 0$ at all times. Now, let us first assume that ξ and d converge to zero but that η and hence ϕ do not. In that case, when ξ converges to zero, we must have $\psi_1 = 1$, $\psi_2 = 0$. Furthermore, the dynamics of η is given by

$$\dot{\eta} = \frac{c_1}{\psi_1^2 + \xi^2}\psi_1\psi_2 = \frac{\frac{1}{2}c_1}{\psi_1^2 + \xi^2}\sin(2\eta - 2\phi) \tag{21}$$

And, when ξ converges to zero, $\phi = \phi_{\mathrm{a}} = (2/b)\eta$, and the above equation will become

$$\dot{\eta} = -\frac{c_1}{2}\mu(\eta) \tag{22}$$

where $\mu(\eta) = \sin(2((2-b)/b)\eta)$. Note that, since $-\pi < 2(2 - b/b)\eta < \pi$, $0 < \eta\mu(\eta) \leqslant |\eta|$ for all $\eta \neq 0$. Now consider the Lyapunov function

$$V_0 = \tfrac{1}{2}\eta^2 \tag{23}$$

which is positive definite. The time-derivative of V_0 along the dynamics of η is given by

$$\dot{V}_0 = -\frac{c_1}{2}\eta\mu(\eta) \leqslant -\frac{c_1}{2}|\eta| \tag{24}$$

which is negative definite. This implies that V_0, and hence, η are bounded everywhere. Moreover, η must exponentially converge to the largest invariant set in $D_1 \cup \{0\}$. Hence, according to Eq. (7) ϕ also must converge to the largest invariant set in $B \cup \{0\}$ where $B = \{\phi\colon p \in D\} \subseteq D$. In other words, from all the above, the state vector p must converge to the largest invariant set in $D \cup \{0\}$. We are now going to show that the largest invariant set in $D \cup \{0\}$ is the origin. Note that the points on the discontinuity surfaces D_1, D_2 and D_3 can be denoted by $m_1 = 2\eta - s_\eta\pi = 0$, $m_2 = \phi - s_\phi\pi = 0$ and $m_3 = a\xi - s_\xi\pi = 0$, respectively, where $s_x = \mathrm{sign}(x)$. The dynamics of these variables are then given by

$$\dot{m}_1 = \frac{c_1}{\psi_1^2 + \xi^2}\sin(2\phi - m_1) \tag{25}$$

$$\dot{m}_2 = \left(c_2 + \frac{c_1 d^2}{\psi_1^2 + \xi^2}\right)\xi + \frac{\frac{c_1}{b}}{\psi_1^2 + \xi^2}\sin(2\eta - 2m_2) \tag{26}$$

$$\dot{m}_3 = -\left(c_2 + c_1\frac{d^2 + \frac{b-1}{x}\left|\sin\left(2\frac{2-b}{b}\eta\right)\right|}{\psi_1^2 + \xi^2}\right)(m_3 + s_\xi\pi) \tag{27}$$

For the discontinuity surface D to be invariant, either m_i and $\dot{m}_i$ must be zero, for $i = 1, 2, 3$, or $m_i\dot{m}_i < 0$ on both sides of D_i, indicating high-frequency limit cycles. Now assume that m_1 and $\dot{m}_1$ are both zero. Then, from Eq. (25), we must have $\phi = k(\pi/2)$, for some fixed integer k. However, in that case, the dynamics of ϕ will be given by

$$\dot{\phi} = \left(c_2 + \frac{c_1 d^2}{\psi_1^2 + \xi^2}\right)\xi \tag{28}$$

which, since $\xi = ((2s_\eta/b) - k)(\pi/2) \neq 0$, it must be nonzero at all times. Therefore, ϕ, $\dot{m}_1$ and consequently m_1 must change. Also, from Eq. (25), $m_1\dot{m}_1 \not< 0$ on both sides of D_1 except when $\phi = k\pi$. But, since we have already shown that ϕ must change away from any value of $k(\pi/2)$, this means that $m_1\dot{m}_1$ cannot be negative on both sides of D_1 at all times. These observations imply that D_1 is not an invariant set. Now assume that m_2 and $\dot{m}_2$ are both zero. Then, from Eq. (26), we must have

$$\sin(2\eta) = -\frac{\psi_1^2 + \xi^2}{\frac{c_1}{b}}\left(c_2 + \frac{c_1 d^2}{\psi_1^2 + \xi^2}\right)\xi.$$

However, in that case, the dynamics of η will be given by

$$\dot{\eta} = \frac{\frac{c_1}{2}}{\psi_1^2 + \xi^2}\sin 2\eta = -\frac{b}{2}\left(c_2 + \frac{c_1 d^2}{\psi_1^2 + \xi^2}\right)\xi \tag{29}$$

But, since $m_2 = 0$, $\xi = (2/b)\eta - s_\phi \pi$ modulo the interval $(1/a)(-\pi, \pi]$, and that $(\pi/2) < (2/b)|\eta| < \pi$, $\dot{\eta}$ must be nonzero at all times. This implies that η, $\dot{m}_2$ and m_2 must change. Also, since $\xi = (2/b)\eta - m_2 - s_\phi \pi$ modulo the interval $(1/a)(-\pi, \pi]$, from Eq. (26), $m_2\dot{m}_2$ cannot be negative on both sides of D_2 for any value of η. All these imply that D_2 is not an invariant set. In addition, Eq. (27) can never be made equal to zero and, since $s_{m_3} = -s_\xi$, $m_3\dot{m}_3$ cannot be negative on both sides of D_3. These imply that D_3 is not an invariant set either. From all the above, we can now state that D is not an invariant set. Therefore, the state vectors p, and equivalently q, must exponentially converge to zero. Moreover, since V is radially unbounded, the origin is a globally exponentially stable equilibrium point of the closed-loop system [14,15].

In addition, d, ξ, ϕ and η are continuous everywhere except on the discontinuity surface D. Therefore, functions ψ_1 and ψ_2 must be continuous everywhere except on the discontinuity surface D. This in turn implies that the control variables v and ω must also be continuous everywhere except on the discontinuity surface D. But, the discontinuity surface D is not an invariant set and, hence, the robot cannot get stuck in D. Therefore, the robot trajectory can have at most a finite number of discontinuities and, hence, the control variables v and ω are piecewise continuous. Moreover, since d, ξ, ψ_1 and ψ_2 are bounded, the control variables v and ω must also be bounded.

5. Simulation results

Here, computer simulation results are presented to show the effectiveness of the proposed controller. The proposed controller is implemented, as shown in Fig. 3, to automatically steer a wheeled mobile robot from any nonzero initial position and orientation to the target equilibrium state $q_d = [0, 0, 0]^T$. The simulations are carried out using the MATLAB/SIMULINK software, and the results are shown in Figs. 4–7. The control gains are chosen as $c_1 = c_2 = 5$.

Figs. 4 and 5 show, respectively, the time history of the generalized position vector $q = [x, y, \phi]^T$ (position and orientation) and the equivalent transformed state $p = [d, \xi, \phi]^T$ for the robot, where the robot's initial position/orientation is $q(0) = [-1, -2, 0]^T$. Both vectors converge to zero exponentially, as they should. Controller gains c_1 and c_2 determine the rate of convergence of these vectors to zero. Higher values of c_1 and c_2 result in faster convergence.

Figs. 6 and 7 show, respectively, the control vector $u = [v, \omega]^T$ and the trajectory of the mobile robot in the plane.

In this case, both control variables, v and ω, are continuous and bounded and the controller successfully steers the robot to its target position/orientation. The piecewise continuity of the

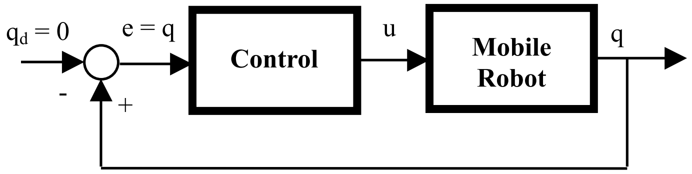

Fig. 3. Block diagram of the simulation example.

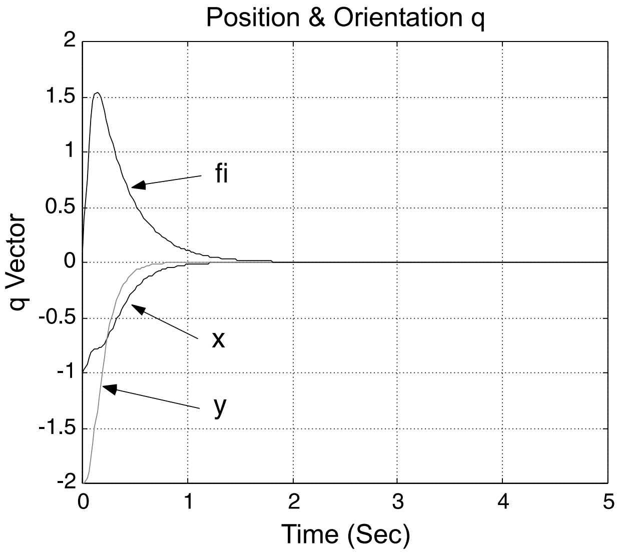

Fig. 4. Position and orientation error vector $q = [x, y, \phi]^{\mathrm{T}}$.

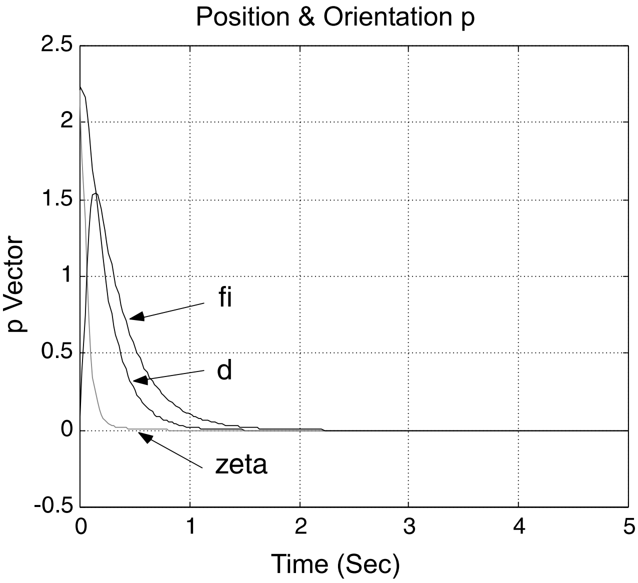

Fig. 5. Equivalent state error vector $p = [d, \xi, \phi]^{\mathrm{T}}$.

control signals is necessary, in practice, for implementing a stabilizing controller when the dynamics of the motors can no longer be ignored.

Also, Fig. 8 shows the robot's trajectory in the plane for various initial positions and orientations. The arrows in the figure indicate the heading (orientation) of the robot at the given point.

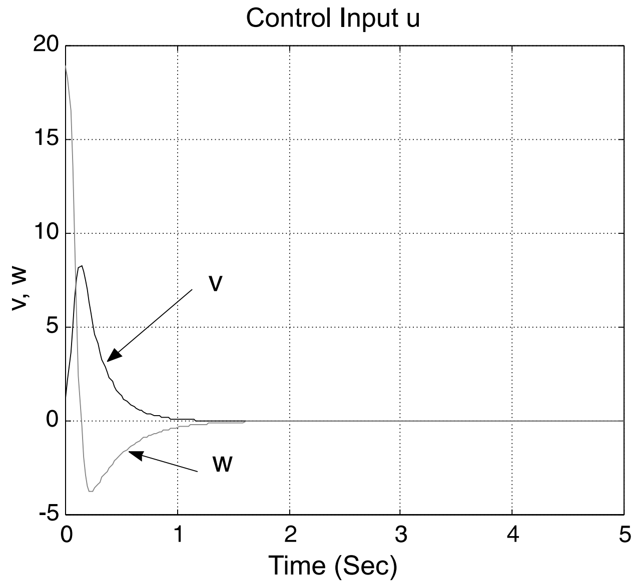

Fig. 6. Control vector $u = [v, \omega]^{\mathrm{T}}$.

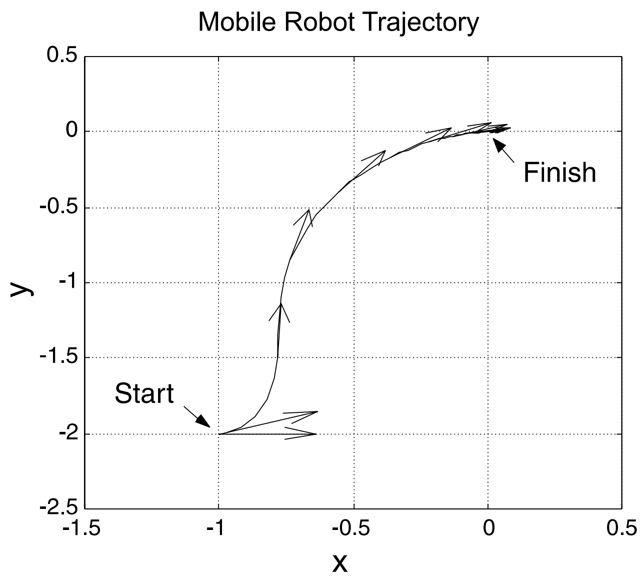

Fig. 7. Robot trajectory in x–y plane with initial condition $q(0) = [-1, -2, 0]^{\mathrm{T}}$.

In all cases, the controller successfully drives the robot to the origin with desired orientation angle of zero. When necessary the robot backs up to adjust its orientation angle before heading for the final position. It can also be seen from the figure that, at most, there are only a finite

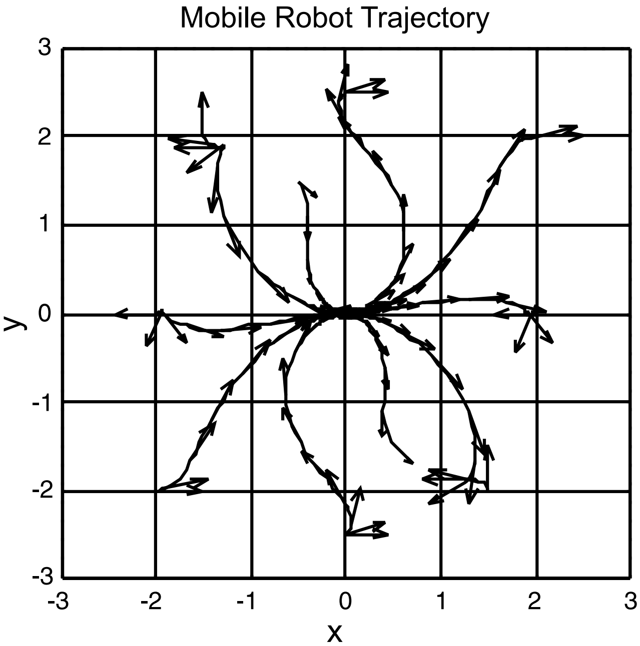

Fig. 8. Robot trajectories in x–y plane, starting from various initial conditions (position and orientation) and reaching the origin while pointing in positive x-direction.

number of points where the robot trajectory is not smooth. These few points are where the robot crosses the discontinuity surface D.

6. Conclusions

In this paper, the problem of stabilizing controller design for point-to-point control of wheeled nonholonomic mobile robots is considered. A discontinuous transformation is used to describe the kinematic model of the system in a form suitable for the controller design. A stabilizing controller is designed that commands the robot to follow an auxiliary direction. However, as the robot gets closer to the target, the auxiliary direction approaches the target orientation. The controller then drives the robot on a trajectory that converges to the desired target position with desired orientation. Computer simulation results favorably support the analytical developments.

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