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Mathematics of Control, Signals, and Systems

Global Asymptotic Stabilization for Controllable Systems without Drift*

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Abstract. This paper proves that the accessibility rank condition on $\mathbb{R}^n\backslash\{0\}$ is sufficient to guarantee the existence of a global smooth time-varying (but periodic) feedback stabilizer, for systems without drift. This implies a general result on the smooth stabilization of nonholonomic mechanical systems, which are generically not smoothly stabilizable using time-invariant feedback.

Key words. Asymptotic stabilization, Controllability, Time-varying feedback.

1. Introduction

Let $f = \{f_1, \ldots, f_m\}$ be $m \geq 2$ vector fields of class C^∞ on $\mathbb{R}^n$. We consider the control system

$$\Sigma: \dot{x} = \sum_{i=1}^{m} u_i f_i(x).$$

Let $\mathrm{Lie}(f)$ be the Lie algebra of vector fields generated by f. Throughout the paper we assume

$$\mathrm{Lie}(f)(x) := \{h(x); h \in \mathrm{Lie}(f)\} = \mathbb{R}^n \qquad \text{for all} \quad x \in \mathbb{R}^n\backslash\{0\}. \tag{1.1}$$

Our main goal is to prove

Theorem 1.1. *For any positive T there exists a feedback law $u = (u_1, \ldots, u_m)$ in $C^\infty(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m)$ such that*:

$$u(0, t) = 0 \textit{ for all } t \textit{ in } \mathbb{R}; \tag{1.2}$$

$$u(x, t + T) = u(x, t) \textit{ for all } x \textit{ in } \mathbb{R}^n \textit{ and } t \textit{ in } \mathbb{R}; \tag{1.3}$$

the origin (of $\mathbb{R}^n$) is a globally asymptotically stable point of

$$\dot{x} = \sum_{i=1}^{m} u_i(x, t) f_i(x). \tag{1.4}$$

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Remark 1.2. (a) It follows from Chow's theorem that (1.1) implies,—and is, in fact, equivalent if f_i is analytic for all i in $[1, m]$—that Σ is completely controllable on $\mathbb{R}^n\setminus\{0\}$.

(b) Brockett proved in [B] that condition (1.1), even with $\mathbb{R}^n$ instead of $\mathbb{R}^n\setminus\{0\}$, does not imply that Σ can be locally asymptotically stabilized by means of a continuous feedback law independent of t, $u = u(x)$. For example, the following control system in $\mathbb{R}^3$

$$\dot{x}_1 = u_1, \qquad \dot{x}_2 = u_2, \qquad \dot{x}_3 = x_1 u_2 - x_2 u_1, \tag{1.5}$$

satisfies (1.1)—even with $\mathbb{R}^3$ instead of $\mathbb{R}^3\setminus\{0\}$—but it is proved in [B] that system (1.5) cannot be locally asymptotically stabilized by a continuous $u = u(x)$. Previous to our work, Samson proved in [S1] that system (1.5) can be globally asymptotically stabilized with a periodic time-varying feedback law. This was the starting point of our study of asymptotic stabilization of systems Σ by means of time-varying feedback laws. Recently, Sepulchre generalized, with different methods, Samson's example. He proved in [S2] that if $m = n - 1$, $\operatorname{rank}\{(f_i(0)); 1 \le i \le n - 1\} = n - 1$, and $\operatorname{Lie}(f)(0) = \mathbb{R}^n$, then system Σ can be locally asymptotically stabilized by means of a periodic time-varying feedback law (which is given explicitly). Finally, we mention that recently Pomet [P] gave an interesting proof of Theorem 1.1 when the following property holds:

$$\operatorname{Span}\{\operatorname{ad}^i_{f_1} f_j(x); 0 \le i, 1 \le j \le m\} = \mathbb{R}^n. \tag{1.6}$$

Pomet also uses our idea to start with a "good" T-periodic time-varying feedback law $\bar{u}$. By "good" we mean (see Section 2) that if $dx/dt = \bar{u}(x, t)f(x)$, then $x(0) = x(T)$ and that the linearized control system along $\dot{x} = \bar{u}(x, t)f(x)$ with $x(0) \neq 0$ is controllable on $[0, T]$. But he introduces a new method to compute a stabilizing feedback law u from this $\bar{u}$. His method, based on the classical Jurdjevic–Quinn approach (see, e.g., [S5, exercise (4.8.1)]), has the advantage of giving u from $\bar{u}$ by means of easier computations than those we propose here and it provides a Lyapunov function (which is useful, see Section 6). The existence of a "good" $\bar{u}$, which is trivial if (1.6) is satisfied, is the main step of our proof. Moreover, a "good" $\bar{u}$ allows the use of the method due to Pomet even if (1.6) is not satisfied (see [CP]).

(c) The interest in time-varying feedback laws for global asymptotic stabilization of (one-dimensional) systems with drift had already been noted in 1980 by Sontag and Sussmann [SS]. Notice that it follows from the proof of [SS, (3.5)] that, with the notations of [SS], if $\pi(0) = \mathbb{R}\setminus\{0\}$, then, for any positive T, the stabilizing time-varying feedback law K can be chosen T-periodic in t; this was not shown in that paper, but it can be seen by moving, if necessary, b_j close to a_{j+1} in such a way that $t_2 \in \{T/k; k \in \mathbb{N}\}$.

(d) Many results on continuous feedback stabilization have been obtained recently. Let us just mention two surveys by Sontag on this problem [S4], [S5, Section 4.8], and the references therein.

As a consequence of Theorem 1.1, and of a method introduced by Tsinias in [T], we will also obtain the conclusion that the system

$$\tilde{\Sigma}: \begin{cases} \dot{x} = \sum_{i=1}^{m} y_i f_i(x), \\ \dot{y} = v, \end{cases}$$

can be globally asymptotically stabilized with a time-varying feedback law. More precisely, we will prove in Section 6 the following result.

Corollary 1.3. *For any positive T, there exists a feedback law $v = (v_1, \dots, v_m)$ in $C^\infty(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}; \mathbb{R}^m)$ such that:*

$$v(0, 0, t) = 0 \textit{ for all } t \textit{ in } \mathbb{R}; \tag{1.7}$$

$$v(x, y, t + T) = v(x, y, t) \textit{ for all } (x, y, t) \textit{ in } \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}; \tag{1.8}$$

the origin of $\mathbb{R}^n \times \mathbb{R}^m$ is a globally asymptotically stable point of

$$\dot{x} = \sum_{i=1}^{m} y_i f_i(x), \qquad \dot{y} = v(x, y, t). \tag{1.9}$$

Remark 1.4. (a) In [CAB] Campion, d'Andréa-Novel, and Bastin have shown that, under some natural physical assumptions, and after a partial feedback linearization, the dynamics of nonholonomic mechanical systems can be reduced to the form $\tilde{\Sigma}$.

(b) An explicit time-varying asymptotically stabilizing feedback law was previously given in [CA] for the special case of the system

$$\dot{x}_1 = y_1 \cos x_3, \qquad \dot{x}_2 = y_1 \sin x_3, \qquad \dot{x}_3 = y_2, \qquad \dot{y}_1 = v_1 \qquad \text{and} \qquad \dot{y}_2 = v_2.$$

These equations describe, e.g., the motion of a unicycle or car in the plane, where the controls v_1 and v_2 are, respectively, the forward acceleration and the angular acceleration (torque) of the steering wheel.

2. The Strategy of the Proof of Theorem 1.1

Let $v \in C^\infty(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m)$ be T-periodic in t, i.e.,

$$v(x_0, t + T) = v(x_0, t) \qquad \text{for all} \quad (x_0, t) \text{ in } \mathbb{R}^n \times \mathbb{R}. \tag{2.1}$$

We assume that v satisfies the following oddness property:

$$v(x_0, T - t) = -v(x_0, t) \qquad \text{for all} \quad (x_0, t) \text{ in } \mathbb{R}^n \times \mathbb{R}, \tag{2.2}$$

and that

$$v(0, t) = 0 \qquad \text{for all } t \text{ in } \mathbb{R}. \tag{2.3}$$

Assume that $\bar{x}$: $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a solution of

$$\frac{\partial \bar{x}}{\partial t} = \sum_{i=1}^{m} v_i(x_0, t) f_i(\bar{x}) = v(x_0, t) f(\bar{x}) \tag{2.4}$$

and

$$\bar{x}(x_0, 0) = x_0 \qquad \text{for all } x_0 \text{ in } \mathbb{R}^n. \tag{2.5}$$

It follows from (2.2) and (2.4) that $t \mapsto \bar{x}(x_0, T - t)$ is also a solution of (2.4); this solution is equal to $\bar{x}(x_0, t)$ for $t = T/2$ and therefore for all t in $\mathbb{R}$; in particular (by taking $t = 0$ and by (2.5)),

$$\bar{x}(x_0, T) = x_0. \tag{2.6}$$

Moreover, if v is "small enough," then x_0 can be smoothly expressed in terms of $\bar{x}(x_0, t)$ and t for all t in $[0, T]$ (see Lemma 3.2). Therefore (2.4) can be written in the following form:

$$\frac{\partial \bar{x}}{\partial t} = \bar{u}(\bar{x}, t) f(\bar{x}), \tag{2.7}$$

where $\bar{u} \in C^\infty(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m)$ and is T-periodic (see (2.6)). The time-varying feedback control law $\bar{u}$ does not make the origin an asymptotically stable equilibrium; indeed, all the trajectories of $\dot{x} = \bar{u}(x, t)f(x)$ are T-periodic (see 2.6). The idea is to try to introduce some dissipation by slightly perturbing v: $v^\varepsilon = v + \varepsilon w$. We will see that if v and w are appropriately chosen ($v = 0$ is not enough) then the time-varying feedback law u^ε, which is associated with v^ε in the same manner as $\bar{u}$ is associated with v, makes, if ε is small enough but positive, the origin a globally asymptotically stable equilibrium. The proof will rely on an analysis of the linearized equation associated with (2.4). Precisely, let $\bar{x}^\varepsilon = \bar{x} + \varepsilon y + O(\varepsilon^2)$ be the solution of (2.4) and (2.5) with v^ε used instead of v. We would like to have, for example,

$$y(x_0, T) = -\eta(x_0)x_0, \tag{2.8}$$

where $\eta(x_0)$ is small but positive whenever $x_0 \neq 0$. The equations satisfied by y are

$$\frac{\partial y}{\partial t} = w(x_0, t)f(\bar{x}) + v(x_0, t)\frac{\partial f}{\partial x}(\bar{x})y \tag{2.9}$$

and

$$y(x_0, 0) = 0. \tag{2.10}$$

Let δ be in $(0, T/8)$. We would like to design $w(x_0, t)$ so that it connects in the interval $[(3T/4) - \delta, (3T/4) + \delta]$ the following two open loop solutions y^* and y_* of (2.9):

$$\frac{\partial y^*}{\partial t} = v(x_0, t)\frac{\partial f}{\partial x}(\bar{x})y^*, \qquad y^*(x_0, T) = -\eta(x_0)x_0, \tag{2.11}$$

$$y_* \equiv 0. \tag{2.12}$$

To show that such a connection is possible, let $d \in C^\infty([0, T]; [0, 1])$ be such that

$$d(t) = 0 \qquad \text{for} \quad t \leq (3T/4) - (\delta/2) \tag{2.13}$$

and

$$d(t) = 1 \qquad \text{for} \quad t \geq (3T/4) + (\delta/2). \tag{2.14}$$

Let us define $z \in C^\infty(\mathbb{R}^n \times [0, T]; \mathbb{R}^n)$ by

$$y(x_0, t) = (1 - d(t))y_*(x_0, t) + d(t)y^*(x_0, t) + z(x_0, t). \tag{2.15}$$

From (2.9) and (2.15) we get

$$\frac{\partial z}{\partial t} - w(x_0, t)f(\bar{x}) - v(x_0, t)\frac{\partial f}{\partial x}(\bar{x})z = r(x_0, t) \tag{2.16}$$

with

$$r(x_0, t) = d(t)(y^*(x_0, t) - y_*(x_0, t)). \tag{2.17}$$

Let us remark that, by (2.13) and (2.14),

$$r = 0 \quad \text{on } \mathbb{R}^n \times ([0, (3T/4) - (\delta/2)] \cup [(3T/4) + (\delta/2), T]). \tag{2.18}$$

Now we remark that if the linear time-varying system

$$\dot{z} = w(x_0, t)f(\bar{x}) + v(x_0, t)\frac{\partial f}{\partial x}(\bar{x})z, \tag{2.19}$$

where w is the control, is controllable with "impulsive" controls (or is uniformly controllable in the sense given by Silverman and Meadows in [SM]), then, according to a theorem due to Ilchmann, Nürnberger, and Schmale [INS; theorem 6.4] (see also [G, 2.3.8(B)]), (2.16) has an "algebraic" solution, i.e., there exist an integer S and mappings (μ_i; $0 \leq i \leq S$) and v_i; $1 \leq i \leq S$) with values into $\mathscr{L}(\mathbb{R}^n; \mathbb{R}^n)$ and $\mathscr{L}(\mathbb{R}^n; \mathbb{R}^m)$, respectively, such that, if

$$z = \sum_{i=0}^{S} \mu_i(x_0, t)\frac{\partial^i r}{\partial t^i}, \tag{2.20}$$

$$w = \sum_{i=0}^{S} v_i(x_0, t)\frac{\partial^i r}{\partial t^i}, \tag{2.21}$$

then (2.16) is satisfied. Let us remark that the importance of the algebraic techniques presented in [G, 2.3.8(B)] has already been noticed in [G, p. 184] for a problem that is related to ours. We will prove in Section 4 that, for appropriate v (generic "small" v are in fact good, see [C], but not $v = 0$; see also Remark 4.2(b)), (2.19) is controllable with impulsive controls (on $[(3T/4) - \delta, (3T/4) + \delta]$); in fact, proceeding as in [INS] or in [G, 2.3.8(B)], we will establish directly the existence of S, (μ_i; $1 \leq i \leq S$), and (v_i; $1 \leq i \leq S$).

An important consequence of (2.18), (2.20), and (2.21) is

$$z = 0, \quad w = 0 \quad \text{on } \mathbb{R}^n \times ([0, (3T/4) - (\delta/2)] \cup [(3T/4) + (\delta/2), T]). \tag{2.22}$$

Let x^ε: $\mathbb{R}^n \times [0, T] \to \mathbb{R}^n$ be such that $x^\varepsilon(\cdot, t)$ is the inverse of $\bar{x}^\varepsilon(\cdot, t)$, see Lemma 3.2, i.e.,

$$\bar{x}^\varepsilon(x^\varepsilon(x, t), t) = x \qquad \text{for all} \quad (x, t) \text{ in } \mathbb{R}^n \times [0, T], \tag{2.23}$$

and let u^ε be the closed-loop control

$$u^\varepsilon(x, t) = v^\varepsilon(x^\varepsilon(x, t), t). \tag{2.24}$$

In order to have a smooth T-periodic u^ε we will require that v satisfy

$$v(x_0, t) = 0 \qquad \text{for all} \quad (x_0, t) \text{ in } \mathbb{R}^n \times \left(\bigcup_{k \in \mathbb{Z}} [kT - (T/8), kT + (T/8)]\right); \tag{2.25}$$

indeed using (2.22), (2.24), and (2.25) we get, since $\delta \in (0, T/8)$,

$$u^\varepsilon = 0 \quad \text{on } \mathbb{R}^n \times ([0, T/8] \cup [7T/8, T]) \tag{2.26}$$

and, therefore, if we extend u^ε to all $\mathbb{R}^n \times \mathbb{R}$ by requiring that u^ε be T-periodic in t, then u^ε is smooth. Moreover, if P: $\mathbb{R}^n \to \mathbb{R}^n$ is the map which associates to x_0 the value of φ at time T, where φ is defined by

$$\dot{\varphi} = u^\varepsilon(\varphi) f(\varphi), \qquad \varphi(0) = x_0, \tag{2.27}$$

then, as $\varepsilon \to 0$,

$$P(x_0) = \bar{x}^\varepsilon(x_0, T) \simeq (1 - \varepsilon \eta(x_0)) x_0; \tag{2.28}$$

hence we expect that the zero state in $\mathbb{R}^n$ becomes globally asymptotically stable for $\dot{x} = u^\varepsilon(x) f(x)$, if ε is small enough but positive. In Section 5 we will prove this global asymptotic stability.

Let us mention that our proof is strongly related to Sontag's paper [S3]. In our context, [S3] gives the existence of nonsingular trajectories of Σ joining x_0 (in $\mathbb{R}^n \setminus \{0\}$) to itself—nonsingular means that the time-varying linear system obtained by linearizing along the trajectory is completely controllable. But it does not seem to follow directly from Sontag's method that there exist such trajectories which depend *smoothly* on x_0 on *all* $\mathbb{R}^n \setminus \{0\}$. In order to take care of this problem, instead of Sard's theorem as in [S3], we have used [G, 2.3.8 E)], where Gromov proves that generic (partial) differential underdetermined linear systems have an algebraic inverse. This has also the advantage of producing a more explicit stabilizing feedback law.

Let us finally remark that, if in our proof we start by studying an open-loop problem (v depends on the initial data), what we obtain at the end is a feedback law (u^ε) which does not depend on the initial data; this is a memoryless control on the augmented state space $\mathbb{R}^n \times (\mathbb{R}/T\mathbb{Z})$.

3. Technical Results

In order to simplify somewhat our proof we first observe:

Lemma 3.1. *Without loss of generality we may assume that, for any i in $[1, m]$,*

$$\|f_i\|_\infty + \sum_{\alpha=1}^{n} \left\| \frac{\partial f_i}{\partial x_\alpha} \right\|_\infty + \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \left\| \frac{\partial^2 f_i}{\partial x_\alpha \, \partial x_\beta} \right\|_\infty \leq 1, \tag{3.1}$$

where $\|h\|_\infty$ denotes $\sup\{|h(x)|;\, x \in \mathbb{R}^n\}$.

Proof. In order to apply Lemma A.1 in Appendix A we let

$$\psi_0(x) = \sum_{i=1}^{m} |f_i(x)| + \sum_{i=1}^{m} \sum_{\alpha=1}^{n} \left| \frac{\partial f_i}{\partial x_\alpha}(x) \right| + \sum_{i=1}^{m} \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \left| \frac{\partial^2 f_i}{\partial x_\alpha \, \partial x_\beta}(x) \right|$$

and

$$\psi_j(x) = 0 \qquad \text{if} \quad j \in \mathbb{N} \setminus \{0\}.$$

It follows from Lemma A.1 that there exists a function θ in $C^\infty(\mathbb{R}^n; [0, +\infty))$ such that $\theta > 0$ on $\mathbb{R}^n \setminus \{0\}$ and, if $\bar{f}_i = \theta f_i$, then, for all i in $[1, m]$,

$$\|\bar{f}_i\|_\infty + \sum_{\alpha=1}^{n} \left\| \frac{\partial \bar{f}_i}{\partial x_\alpha} \right\|_\infty + \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \left\| \frac{\partial^2 \bar{f}_i}{\partial x_\alpha \, \partial x_\beta} \right\|_\infty < +\infty. \tag{3.2}$$

Hence, by replacing f_i by $\lambda \bar{f}_i$ where $\lambda \in (0, +\infty)$ is small enough, inequality (3.1) is satisfied. Clearly (1.1) still holds for $\lambda \bar{f}_i$. Moreover, if a feedback u which globally asymptotically stabilizes the transformed system $\dot{x} = u\lambda \bar{f}$, then, after multiplication by $\lambda\theta$, it also provides a feedback which globally asymptotically stabilizes the original system Σ. ■

From now on we assume that (3.1) holds. Our next lemma gives estimates on $\bar{x}(x_0, t) - x_0$.

Lemma 3.2. *Let v in $C^\infty(\mathbb{R}^n \times [0, T]; \mathbb{R}^m)$ satisfy* (2.3)—*we do not assume* (2.2). *Then, on $\mathbb{R}^n \times [0, T]$ and for some positive constant C_0 independent of v,*

$$\left\| \frac{\partial \bar{x}}{\partial x_0} - \mathrm{Id} \right\| \leq C_0 \left(\|v\|_\infty + \left\| \frac{\partial v}{\partial x_0} \right\|_\infty \right), \tag{3.3}$$

where Id *denotes the identity map from $\mathbb{R}^n$ into $\mathbb{R}^n$, and*

$$|\bar{x} - x_0| \leq C_0 \left(\|v\|_\infty + \left\| \frac{\partial v}{\partial x_0} \right\|_\infty \right) \min\{1, |x_0|\}. \tag{3.4}$$

Before giving a proof of Lemma 3.2 we remark from (3.3) that, for $\|v\|_\infty + \|\partial v/\partial x\|_\infty$ less than $1/C_0$ and for any t in $[0, T]$, $x_0 \to \bar{x}(t, x_0)$ is a diffeomorphism of $\mathbb{R}^n$. It follows that the map $x^0 \colon \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$ defined by $\bar{x}(x^0(x, t), t) = x$ is in $C^\infty(\mathbb{R}^n \times [0, T]; \mathbb{R}^n)$. Moreover, a useful consequence of (3.4) is that, for $\|v\|_\infty + \|\partial v/\partial x\|_\infty$ less than $(2C_0)^{-1}$,

$$|x_0|/2 \leq |\bar{x}| \leq |x_0| + 1 \quad \text{on } \mathbb{R}^n \times [0, T]. \tag{3.5}$$

Proof of Lemma 3.2. Differentiating (2.4) and (2.5) with respect to x_0 we get

$$\frac{\partial}{\partial t}\left(\frac{\partial \bar{x}}{\partial x_0} \right) = \frac{\partial v}{\partial x_0} f(\bar{x}) + v \frac{\partial f}{\partial x}(\bar{x}) \frac{\partial \bar{x}}{\partial x_0}, \tag{3.6}$$

$$\frac{\partial \bar{x}}{\partial x_0}(x_0, 0) = \mathrm{Id}. \tag{3.7}$$

Inequality (3.3) follows from (3.1), (3.6), and (3.7). In a similar way, (3.4) is a consequence of (2.3), (2.4), (2.5), (3.1), and (3.3). ■

4. Algebraic Inverse of the Linearized Equation

Our goal in this section is to prove that, for some v, (2.16) has a solution of the form given by (2.20) and (2.21). Let us first notice that (2.16) contains n equations with

$n + m$ unknown (z_α with $1 \le \alpha \le n$ and w_j with $1 \le j \le m$). It is proved in [G, 2.3.8] that generic (partial) differential underdetermined linear systems have algebraic inverses. Of course, system (2.16) is not generic but the techniques used in [G, 2.3.8] will help us in finding an algebraic solution to (2.16).

To make the analysis simpler we choose v as follows:

$$v(x_0, t) = a(x_0)b(t), \tag{4.1}$$

where the maps $b \in C^\infty(\mathbb{R}; \mathbb{R}^m)$ and $a \in C^\infty(\mathbb{R}^n; \mathbb{R})$ satisfy:

$$b(t + T) = b(t) \qquad \text{for all } t \text{ in } \mathbb{R}; \tag{4.2}$$

$$b(T - t) = -b(t) \qquad \text{for all } t \text{ in } \mathbb{R}; \tag{4.3}$$

$$b = 0 \qquad \text{on} \quad [0, T/8] \cup [7T/8, T]; \tag{4.4}$$

$$a(0) = 0, \qquad a > 0 \qquad \text{on } \mathbb{R}^n \backslash \{0\}. \tag{4.5}$$

Now, given δ in $(0, T/8)$, we define the set

$$Q = \mathbb{R}^n \times [(3T/4) - \delta, (3T/4) + \delta], \tag{4.6}$$

and the operators $\tilde{L}\colon C^\infty(Q; \mathbb{R}^n) \times C^\infty(Q; \mathbb{R}^m) \to C^\infty(Q; \mathbb{R}^n)$ and $L\colon C^\infty(Q; \mathbb{R}^n) \to C^\infty(Q, \mathbb{R}^n)$ by

$$\tilde{L}(z, w) = \frac{\partial z}{\partial t} - wf(\bar{x}) - v\frac{\partial f}{\partial x}(\bar{x})z, \tag{4.7}$$

$$L(z) = \tilde{L}(z, 0). \tag{4.8}$$

Note that $\tilde{L}(z, w) = r$ is the linearized equation (see (2.16)).

We shall also need some combinatorial notations. Let $\mathscr{E}_k$ be the set of sequences $I = i_1 i_2 \cdots i_k$ of k elements from $\{1, 2, \dots, m\}$; the length k of the sequence I will be denoted by $|I|$. For convenience, we denote by $\mathscr{E}_0$ the set whose unique element is the empty sequence denoted by $\varnothing$; we have $|\varnothing| = 0$. For $I = i_1 i_2 \cdots i_k$ and $J = j_1 j_2 \cdots j_{k'}$, we define $I * J \in \mathscr{E}_{k+k'}$ by

$$I * J = i_1 i_2 \cdots i_k j_1 j_2 \cdots j_{k'}.$$

Let $\mathscr{E} = \bigcup_{k \ge 0} \mathscr{E}_k$. For $I \in \mathscr{E}\backslash\{\varnothing\}$ we define f_I by induction on $|I|$ in $C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ by requiring

$$f_{i*J} = [f_i, f_J] \qquad \text{for all } J \text{ in } \mathscr{E}\backslash\{\varnothing\} \text{ and all } i \text{ in } [1, m]. \tag{4.9}$$

Note that f_I is already defined if $|I| = 1$. For an integer l, let $\mathrm{Lie}_l(f)$ be the vector subspace of $\mathrm{Lie}(f)$ generated by iterated Lie brackets containing l or fewer vectors in $\{f_1, \dots, f_m\}$. Using Jacobi's identity we easily check that, in the vector space of smooth tangent vector fields on $\mathbb{R}^n$,

$$\mathrm{Lie}_l(f) = \mathrm{Span}\{f_I; 0 < |I| \le l\}. \tag{4.10}$$

To make the arguments clearer we will first start with the case where we have, instead of (1.1), the stronger hypothesis:

there are s vector fields $X_1, \dots, X_s$ in $\mathrm{Lie}(f)$ such that $X_1(x), \dots, X_s(x)$ span $\mathbb{R}^n$ for all x in $\mathbb{R}^n\backslash\{0\}$. (4.11)

Then, by (4.10) and (4.11), there exists an integer l such that

$$\mathrm{Span}\{f_I(x); 0 < |I| \le l\} = \mathbb{R}^n \qquad \text{for all } x \text{ in } \mathbb{R}^n \setminus \{0\}. \tag{4.12}$$

With $\bar{x}$ defined by (2.4) and (2.5), let $g_1 = vf(\bar{x})$. By induction we can get for $p \ge 1$, with $g_p = L^{p-1}(g_1)$,

$$g_p(x_0, t) = \sum_{0<|I|\le p} a(x_0)^{|I|} C_p(I)(t) f_I(\bar{x}(x_0, t)), \tag{4.13}$$

where $C_p(I)$ is defined by

$$C_0(\varnothing) = 1, \qquad C_0(I) = 0 \quad \text{if } |I| \ge 1, \qquad C_p(\varnothing) = 0 \quad \text{if } p > 0, \tag{4.14}$$

$$C_p(i * I) = b_i C_{p-1}(I) + C_{p-1}(i * I) \qquad \text{for all} \quad I \text{ in } \mathscr{E}, i \text{ in } [1, m], \text{and } p \ge 1. \tag{4.15}$$

For example, $C_1(i) = b_i$, $C_1(I) = 0$ if $|I| \ge 2$, and $C_2(i) = b_i^{(1)}$, $C_2(i_1 i_2) = b_{i_1} b_{i_2}$. By induction arguments we have

$$C_p(i) = b_i^{(p-1)} \qquad \text{for all } i \text{ in } [1, m] \text{ and all } p \ge 1, \tag{4.16}$$

$$C_p(I) = 0 \qquad \text{if} \quad |I| \ge p + 1, \tag{4.17}$$

and

$C_p(I)$ is a homogeneous polynomial of degree $|I|$ in the variables $b_i^{(j)}$ with $1 \le i \le m$ and $j \le p - |I|$. (4.18)

We will see in (4.29) that the system $\tilde{L}(z, w) = kg_p$ (where the data are the functions k and the unknown the maps z and w) is algebraically solvable. Hence our goal is to try to express $f_I(\bar{x})$, for I with $0 < |I| \le l$, in terms of g_p with $p \ge 1$. Until (4.20), and in Appendix B, we consider $b_i^{(j)}$ as formal independent variables. Let R be the field of rational functions in the variables ($b_i^{(j)}$; $i \in [1, m]$, $j \in \mathbb{N}$) and let $q(l) = \sum_{j=1}^{l} q^*(j)$ with $q^*(j) = m(m^j - 1)/(m - 1)$. In Appendix B we prove

Lemma 4.1. *Let, for $0 < |I| \le l$, $A(I)$ be in R and such that, in R,*

$$\sum_{0<|I|\le l} A(I) C_r(I) = 0 \qquad \textit{for all } r \textit{ in } [1, q(l)]. \tag{4.19}$$

Then

$$A(I) = 0 \qquad \textit{for all } I \textit{ with } 0 < |I| \le l.$$

Lemma 4.1 can be rephrased in the following way. Let us introduce an ordering on the I with $0 < |I| \le l$ (e.g., lexicographical) and let us consider the nonsquare $q(l) \times q^*(l)$ matrix with entries $C_r(I)$, $1 \le r \le q(l)$, $0 < |I| \le l$. Then Lemma 4.1 just tells us that this matrix has full column rank $q^*(l)$. Therefore it has a left inverse, i.e., there are elements ($R_p(J)$; $0 < |J| \le l$, $1 \le p \le q(l)$) in R such that for all I, J in $\mathscr{E}$ with $0 < |I| \le l$, $0 < |J| \le l$, we have, in R,

$$\sum_{1 \le p \le q(l)} R_p(I) C_p(J) = 1 \text{ if } I = J, 0 \text{ if } I \ne J. \tag{4.20}$$

We now choose a map $b \in C^\infty(\mathbb{R}; \mathbb{R}^m)$ satisfying (4.2), (4.3), and (4.4) and such that

the denominator of $R_p(I)$ evaluated at $3T/4$ is not zero for all p with $1 \le p \le q(l)$ and all I with $0 < |I| \le l$. (4.21)

By continuity, it follows from (4.17) that there exists a δ in $(0, T/8)$ such that, for $1 \le p \le q(l)$ and $0 < |I| \le l$, the denominator of $R_p(I)$ does not vanish on $[3T/4 - \delta, 3T/4 + \delta]$. Hence each $R_p(I)$ can now be considered as a function in $C^\infty([(3T/4) - \delta, (3T/4) + \delta]; \mathbb{R})$. From (4.20) we get, for all I with $0 < |I| \le l$,

$$f_I(\bar{x}) = \sum_{p=1}^{q(l)} \frac{1}{a^{|I|}} R_p(I) h_p \tag{4.22}$$

with

$$h_p = \sum_{0<|J|\le l} a^{|J|} C_p(J) f_J(\bar{x}). \tag{4.23}$$

Now using (4.13) and (4.23) we get

$$h_p = g_p - \sum_{l<|J|\le p} a^{|J|} f_J(\bar{x}). \tag{4.24}$$

But from (4.12), using a standard partition of unity argument, we get, for $l < |J| \le p$,

$$f_J(\bar{x}) = \sum_{0<|K|\le l} \alpha(J, K)(\bar{x}) f_K(\bar{x}), \tag{4.25}$$

where $\alpha(J, K) \in C^\infty(\mathbb{R}^n \setminus \{0\})$. From (4.22), (4.24), and (4.25) we get, for $|I| \le l$,

$$\begin{aligned} \sum_{1\le p\le q(l)} \sum_{l<|J|\le p} \sum_{0<|K|\le l} & a^{|J|-|I|} R_p(I) C_p(J) \alpha(J, K)(\bar{x}) f_K(\bar{x}) + f_I(\bar{x}) \\ &= \sum_{p=1}^{q(l)} \frac{1}{a^{|I|}} R_p(I) g_p. \end{aligned} \tag{4.26}$$

Hence, using Lemma 3.2, Appendix A, and (4.26) (note that in (4.26) $|J| - |I| > 0$), we get for some function a in $C^\infty(\mathbb{R}^n; [0, +\infty))$, with $\|a\|_\infty + \|\partial a/\partial x\|_\infty$ small, and for I with $0 < |I| \le l$,

$$f_I(\bar{x}(x_0, t)) = \sum_{1\le p\le q(l)} \beta(I, p)(x_0, t) g_p(x_0, t), \tag{4.27}$$

where $\beta(I, p)$ is of class C^∞ on $Q' = Q \setminus (\{0\} \times [3T/4 - \delta, 3T/4 + \delta])$. Moreover, for $k \in C^\infty(Q'; \mathbb{R})$, we have, for $p \ge 2$,

$$L(kg_{p-1}) = \frac{\partial k}{\partial t} g_{p-1} + kg_p \qquad \text{and} \qquad \tilde{L}(0, kv) = -kg_1, \tag{4.28}$$

and therefore, by induction on p,

$$\tilde{L}\left(\sum_{j=0}^{p-2} (-1)^j \left(\frac{\partial^j k}{\partial t^j}\right) g_{p-1-j}, (-1)^p \left(\frac{\partial^{p-1}}{\partial t^{p-1}} k\right) v\right) = kg_p. \tag{4.29}$$

Using (4.12), (4.27), and again a partition of unity we have

$$e_i = \sum_{1\le p\le q(l)} \gamma(i, p) g_p, \qquad 1 \le i \le n, \tag{4.30}$$

where $\gamma(i, p) \in C^\infty(Q')$ and (e_i) is a basis of $\mathbb{R}^n$. Let us denote by $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^q)$ the set of linear maps from $\mathbb{R}^n$ into $\mathbb{R}^q$. It follows from (4.29) and (4.30) that there exist maps

$(\mu_i; 0 \leq i \leq q(l) - 2)$, $(v_j; 0 \leq j \leq q(l) - 1)$ such that:

$$\mu_i \in C^\infty(Q'; \mathscr{L}(\mathbb{R}^n; \mathbb{R}^n)), \tag{4.31}$$

$$v_j \in C^\infty(Q'; \mathscr{L}(\mathbb{R}^n; \mathbb{R}^m)), \tag{4.32}$$

$$\tilde{L}\left(\sum_{i=0}^{q(l)-2} \mu_i \frac{\partial^i r}{\partial t^i}, \sum_{j=0}^{q(l)-1} v_j \frac{\partial^j r}{\partial t^j}\right) = r \text{ for all } r \text{ in } C^\infty(Q'; \mathbb{R}^n). \tag{4.33}$$

Finally, using Corollary A.2, we see that there exists $\eta \in C^\infty(\mathbb{R}^n; [0, +\infty))$ such that

$$\eta(0) = 0, \qquad \eta > 0 \quad \text{on } \mathbb{R}^n \backslash \{0\}, \tag{4.34}$$

and, if r is defined by (2.17), then z and w, defined by

$$z = \sum_{i=0}^{q(l)-2} \mu_i \frac{\partial^i r}{\partial t^i}, \qquad w = \sum_{i=0}^{q(l)-1} v_i \frac{\partial^i r}{\partial t^i}, \tag{4.35}$$

are of class C^∞ on Q and vanish on $Q \cap (\{0\} \times \mathbb{R})$. By (4.33), z and w satisfy (2.16). Let us remark that they also vanish on $Q \backslash (\mathbb{R}^n \times [(3T/4) - (\delta/2), (3T/4) + (\delta/2)])$ (see (2.18) and (4.35)). We extend them to $\mathbb{R}^n \times [0, T]$ by

$$z(x, t) = 0 \qquad \text{and} \qquad w(x, t) = 0 \qquad \text{for all} \quad (x, t) \notin Q. \tag{4.36}$$

Notice that we may take a, b, and w so that

$$\|w\|_\infty + \left\|\frac{\partial w}{\partial x_0}\right\| + \|v\|_\infty + \left\|\frac{\partial v}{\partial x_0}\right\| \leq \frac{1}{2C_0}, \tag{4.37}$$

where C_0 is defined in Lemma 3.2.

Finally, we briefly return to the case where (1.1) holds instead of (4.11). Let $\mathscr{B}$ be set of sequences $(\tau_j; j \in \mathbb{N})$ of elements of $\mathbb{R}^n$. We provide $\mathscr{B}$ with the metric

$$\Delta(\tau, \bar{\tau}) = \sum_{j=0}^{+\infty} \frac{2^{-j}|\tau_j - \bar{\tau}_j|}{1 + |\tau_j - \bar{\tau}_j|}; \tag{4.38}$$

the metric space $(\mathscr{B}, \Delta)$ is complete. A theorem due to Borel (see, e.g., [D, Ex. 4, p. 188]) tells us that for any τ in $\mathscr{B}$ there exists b in $C^\infty(\mathbb{R}; \mathbb{R}^m)$ satisfying (4.2), (4.3), and (4.4) such that

$$b^{(j)}(3T/4) = \tau_j \qquad \text{for all } j \text{ in } \mathbb{N}. \tag{4.39}$$

From this theorem, Lemma 4.1, and Baire's theorem applied to $(\mathscr{B}, \Delta)$, it follows that there exists b in $C^\infty(\mathbb{R}; \mathbb{R}^m)$ satisfying (4.2), (4.3), and (4.4) such that for *all* $l > 0$ (4.21) holds (note that b is universal: it does not depend on f). Next we choose (see, in particular, Appendix A) a satisfying (4.5) such that, for any compact subset K of $\mathbb{R}^n \backslash \{0\}$, (4.27) holds for all x_0 in K but now δ, 1, and β (may) *depend on* K. Using again Appendix A and a partition of unity we get η in $C^\infty(\mathbb{R}^n; [0, +\infty))$ and w in $C^\infty(\mathbb{R}^n \times [0, T]; \mathbb{R}^m)$ satisfying (4.34),

$$w = 0 \qquad \text{on} \quad (\{0\} \times [0, T]) \cup (\mathbb{R}^n \times ([0, T/8] \cup [7T/8, T])) \tag{4.40}$$

and such that y defined by (2.9)–(2.10) satisfies (2.8). Again we may impose (4.37)

Remark 4.2. (a) As mentioned in Section 2, our study of the linearized equation is connected to previous work by Silverman and Meadows [SM]. In particular, it follows from [SM], (4.27), and (4.12) that, for $x_0 \neq 0$, the time-varying linear system

$$\dot{y} = wf(\bar{x}(x_0, \cdot)) + v(x_0, \cdot)\frac{\partial f}{\partial x}(\bar{x}(x_0, \cdot))y,$$

where w is the control, is controllable near $t = 3T/4$ with impulsive controls. This controllability implies the existence of the algebraic inverse by [INS, theorem 6.4].

(b) Let $C_T^\infty(\mathbb{R}; \mathbb{R}^m)$ be the set of functions b in $C^\infty(\mathbb{R}; \mathbb{R}^m)$ satisfying (4.2) and (4.3); $C^\infty(\mathbb{R}; \mathbb{R}^m)$ is equipped with the Whitney topology and $C_T^\infty(\mathbb{R}; \mathbb{R}^m)$ with the induced topology. Let us call b *good* if for any f satisfying (1.1) then, if a is small enough in the $C^\infty(\mathbb{R}^n\backslash\{0\}; (0, +\infty))$ Whitney topology, (2.19) is controllable with impulsive controls at time $t = 3T/4$, for all x_0 in $\mathbb{R}^n\backslash\{0\}$. Then it follows from our proof that generic b are good. Indeed, let $M(l)$ be the set of b such that the matrix $\{C_r(I)(3T/4);\ 0 < |I| \leq l,\ r \leq q(l)\}$ has rank $q^*(l)$. By Lemma 4.1 and Thom's transversality theorem, the open set $M(l)$ is dense in $C_T^\infty(\mathbb{R}; \mathbb{R}^m)$; hence, $\bigcap_{l\geq 1} M(l)$ is residual in the Baire space $C_T^\infty(\mathbb{R}; \mathbb{R}^m)$, but any b in $\bigcap_{l\geq 1} M(l)$ is good. Let us remark that if we require controllability with impulsive controls on all $\mathbb{R}$ instead of just at time $t = 3T/4$, then generic b in $C_T^\infty(\mathbb{R}; \mathbb{R}^m)$ are still good (see [C] for a proof).

5. Study of the Nonlinear Equation

Let us estimate $\bar{x}^\varepsilon(x_0, T)$ where $\bar{x}^\varepsilon$ is defined by

$$\frac{\partial \bar{x}^\varepsilon}{\partial t} = (v + \varepsilon w)f(\bar{x}^\varepsilon) = v^\varepsilon f(\bar{x}^\varepsilon), \tag{5.1}$$

$$\bar{x}^\varepsilon(x_0, 0) = 0. \tag{5.2}$$

We recall, see (2.5) and Sections 2 and 4, that

$$\bar{x}^0(x_0, T) = x_0 \qquad \text{for all } x_0 \text{ in } \mathbb{R}^n, \tag{5.3}$$

$$\left.\frac{\partial \bar{x}^\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0}(x_0, T) = -\eta(x_0)x_0 \qquad \text{for all } x_0 \text{ in } \mathbb{R}^n. \tag{5.4}$$

Differentiating (5.1) and (5.2) with respect to ε we get

$$\frac{\partial}{\partial t}\left(\frac{\partial \bar{x}^\varepsilon}{\partial \varepsilon}\right) = wf(\bar{x}^\varepsilon) + v^\varepsilon \frac{\partial f}{\partial x}(\bar{x}^\varepsilon)\frac{\partial \bar{x}^\varepsilon}{\partial \varepsilon}, \tag{5.5}$$

$$\frac{\partial \bar{x}^\varepsilon}{\partial \varepsilon}(x_0, 0) = 0. \tag{5.6}$$

Going back to Section 4 we see that there exists some function γ in $L^\infty_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\}; (0, +\infty))$ independent of η such that

$$|w(x_0, t)| \leq \gamma(x_0)\eta(x_0). \tag{5.7}$$

From (3.1), (4.37), (5.5), and (5.7) we get, for some constant C_1 independent of η and

ε in $[0, 1]$,

$$\left|\frac{\partial \bar{x}^\varepsilon}{\partial \varepsilon}(x_0, t)\right| \le C_1\left(\gamma(x_0)\eta(x_0) + \left|\frac{\partial \bar{x}^\varepsilon}{\partial \varepsilon}(x_0, t)\right|\right),$$

which, with (5.6), gives, for some constant C again independent of η and ε in $[0, 1]$,

$$\left|\frac{\partial \bar{x}^\varepsilon}{\partial \varepsilon}(x_0, t)\right| \le C\gamma(x_0)\eta(x_0) \qquad \text{for all} \quad (x_0, t) \text{ in } \mathbb{R}^n \times [0, T]. \tag{5.8}$$

Differentiating again (5.5) and (5.6) with respect to ε we have

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 \bar{x}^\varepsilon}{\partial \varepsilon^2}\right) = 2w\frac{\partial f}{\partial x}(\bar{x}^\varepsilon)\frac{\partial \bar{x}^\varepsilon}{\partial \varepsilon} + \left(\frac{\partial^2 f}{\partial x^2}(\bar{x}^\varepsilon)\frac{\partial^2 \bar{x}^\varepsilon}{\partial \varepsilon^2} + \frac{\partial^2 f}{\partial x^2}(\bar{x}^\varepsilon)\left(\frac{\partial \bar{x}^\varepsilon}{\partial \varepsilon}, \frac{\partial \bar{x}^\varepsilon}{\partial \varepsilon}\right)\right), \tag{5.9}$$

$$\frac{\partial^2 \bar{x}^\varepsilon}{\partial \varepsilon^2}(x_0, 0) = 0. \tag{5.10}$$

From (3.1), (5.7), (5.8), (5.9), and (5.10) we have, again for some constant $\bar{C}$ independent of η and ε in $[0, 1]$,

$$\left|\frac{\partial^2 \bar{x}^\varepsilon}{\partial \varepsilon^2}(x_0, T)\right| \le \bar{C}\gamma(x_0)^2\eta(x_0)^2. \tag{5.11}$$

Using Appendix A once more we see that we may also impose on η

$$|x_0|^{-1}\gamma^2\eta \in L^\infty(\mathbb{R}^n). \tag{5.12}$$

By (5.11), (5.12), (5.3), and (5.4) we get for ε in $[0, 1]$ small enough

$$|\bar{x}^\varepsilon(x_0, T)| < \left(1 - \varepsilon\frac{\eta(x_0)}{2}\right)|x_0|. \tag{5.13}$$

Therefore, if we take (see Section 2, Lemma 3.2, and (4.37))

$$u^\varepsilon(x, t) = (v + \varepsilon w)(x^\varepsilon(x, t), t), \tag{5.14}$$

then $u^\varepsilon \in C^\infty(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n)$ satisfies (1.2), (1.3), and (1.4) for ε in $[0, 1]$ small enough (see, in particular, (4.34) and (5.13)).

Remark 5.1. It follows easily from the proof of Theorem 1.1 that assumption (1.1) can be replaced by the following weaker assumption: there exists V in $C^1(\mathbb{R}^n; [0, +\infty))$ with $V(x) \to +\infty$ as $x \to +\infty$ and such that for each x in $\mathbb{R}^n \backslash \{0\}$ there exists h in $\mathrm{Lie}(f)$ satisfying $h(x) \cdot \nabla V(x) \neq 0$. Under this assumption Theorem 1.1 also holds. See [C] for more details.

6. Proof of Corollary 1.3

We proceed as in [T] and [S1, Lemma 4.8.3]. Let u be as in Theorem 1.1. From a classical converse of Lyapunov's second theorem (see [K]) we know that $\dot{x} = uf(x)$ admits a T-periodic Lyapunov function, i.e., there exists $V \in C^\infty(\mathbb{R}^n \times \mathbb{R}; [0, +\infty))$

such that:

$$V(0, t) = 0 \qquad \text{for all } t \text{ in } \mathbb{R}, \tag{6.1}$$

$$V(x, t) > 0 \qquad \text{for all } (x, t) \text{ in } (\mathbb{R}^n \backslash \{0\}) \times \mathbb{R}, \tag{6.2}$$

$$V(x, t + T) = V(x, t) \qquad \text{for all } (x, t) \text{ in } \mathbb{R}^n \times \mathbb{R}, \tag{6.3}$$

$$\lim_{|x| \to +\infty} \min\{V(x, t); t \in [0, T]\} = +\infty, \tag{6.4}$$

$$\frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial x}\right) \cdot (uf(x)) < 0 \quad \text{on } (\mathbb{R}^n \backslash \{0\}) \times \mathbb{R}. \tag{6.5}$$

Let $W\colon \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to [0, +\infty)$ be defined by

$$W(x, y, t) = \tfrac{1}{2}|y - u(x, t)|^2 + V(x, t).$$

Then, by (6.1), (6.2), (6.3), (6.4), (1.2), and (1.3)

$$W(0, 0, t) = 0 \qquad \text{for all } t \text{ in } \mathbb{R}, \tag{6.6}$$

$$W(x, y, t) > 0 \qquad \text{for all } (x, y, t) \text{ in } (\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}) \backslash (\{0\} \times \{0\} \times \mathbb{R}), \tag{6.7}$$

$$W(x, y, t + T) = W(x, y, t) \qquad \text{for all } (x, y, t) \text{ in } \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}, \tag{6.8}$$

$$\lim_{|x| + |y| \to +\infty} \min\{W(x, y, t); t \in [0, T]\} = +\infty. \tag{6.9}$$

Let $v\colon \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ be defined by

$$v_i = -(y - u)_i + \left(\frac{\partial u_i}{\partial x}\right)(y \cdot f) - f_i(x)\frac{\partial V}{\partial x} + \frac{\partial u_i}{\partial t} \qquad \text{for all } i \text{ in } [1, m]. \tag{6.10}$$

From (6.10) we get, for system $\tilde{\Sigma}$

$$\dot{W} = -(y - u)^2 + \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} uf. \tag{6.11}$$

From (6.10), (1.2), (1.3), (6.1), and (6.3) we obtain (1.7) and (1.8). From Lyapunov's second theorem, (6.11), (6.5), (1.2), (6.6), (6.7), (6.8), and (6.9) we get (1.9). ■

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Note Added in Proof. In a recent preprint (Universal nonsingular controls, Rutgers University, December 1991) Sontag has given a different (and shorter) proof of the existence of a good $\bar{u}$ when $f_1, \ldots, f_m$ are analytic.

Appendix A

In this appendix we prove:

Lemma A.1. *Let $(\psi_i; i \in \mathbb{N})$ be a sequence of functions in $L^\infty_{\text{loc}}(\mathbb{R}^n \setminus \{0\}; \mathbb{R})$. Then there exists a function θ in $C^\infty(\mathbb{R}^n; [0, +\infty))$ such that*

$$\theta > 0 \quad \text{on } \mathbb{R}^n \setminus \{0\}, \qquad \theta(0) = 0, \tag{A.1}$$

$$\psi_i \partial^\alpha \theta \in L^\infty(\mathbb{R}^n) \qquad \text{for all } i \text{ in } \mathbb{N} \text{ and for all } \alpha \text{ in } \mathbb{N}^n. \tag{A.2}$$

Proof. Let, for $k \in \mathbb{N}^* \setminus \{0\}$,

$$\omega_k = \{x \in \mathbb{R}^n; k < |x| < k + 2\} \cup \left\{x \in \mathbb{R}^n; \frac{1}{k+2} < |x| < \frac{1}{k}\right\} \tag{A.3}$$

and let

$$\omega_0 = \{x \in \mathbb{R}^n; \tfrac{1}{2} < |x| < 2\}. \tag{A.4}$$

We have

$$\bigcup_{k \in \mathbb{N}} \omega_k = \mathbb{R}^n \setminus \{0\}, \tag{A.5}$$

and

$$\text{for all } x \text{ in } \mathbb{R}^n \text{ there exist at most two indices } k \text{ such that } x \in \omega_k. \tag{A.6}$$

Let $(\theta_k; k \in \mathbb{N})$ be a partition of unity associated with $(\omega_k; k \in \mathbb{N})$, i.e.,

$$\theta_k \in C^\infty(\mathbb{R}^n; [0, 1]), \qquad \text{support } \theta_k \subset \omega_k, \tag{A.7}$$

$$\sum_{k \geq 0} \theta_k(x) = 1 \qquad \text{for all } x \text{ in } \mathbb{R}^n \setminus \{0\}. \tag{A.8}$$

Let c_k be a real number such that

$$0 < c_k, \qquad c_k \sup\{(|\psi_i(x)| + 1) \cdot |\partial^\alpha \theta_k(x)|; x \in \omega_k, \mathrm{i} \leq k, |\alpha| \leq k\} \leq 1/(k + 1), \tag{A.9}$$

and let

$$\theta = \sum_{k \geq 0} c_k \theta_k. \tag{A.10}$$

We easily verify that $\theta \in C^\infty$ $\mathbb{R}^n; [0, +\infty))$ and that it satisfies (A.1) and (A.2). ■

A consequence of Lemma A.1 is

Corollary A.2. *Let $\varphi \in C^\infty((\mathbb{R}^n \setminus \{0\}) \times [0, T]; \mathbb{R}^p)$. Then there exists $\theta \in C^\infty(\mathbb{R}^n; [0, +\infty))$ satisfying* (A.1) *such that*

$$\theta\varphi \text{ extended by } 0 \text{ on } \{0\} \times [0, T] \text{ is in } C^\infty(\mathbb{R}^n \times [0, T]; \mathbb{R}^p), \tag{A.11}$$

$$\partial^\alpha(\theta\varphi) \in L^\infty(\mathbb{R}^n \times [0, T]; \mathbb{R}^p) \text{ for all } \alpha \text{ in } \mathbb{N}^{n+1}. \tag{A.12}$$

Proof. Apply Lemma A.1 with

$$\psi_i(x) = \max\{|\partial^\beta \varphi(x, t)|/\min(|x|, 1);\ t \in [0, T],\ \beta \in \mathbb{N}^{n+1},\ |\beta| \le i\}. \tag{A.13}$$

■

Appendix B

This appendix is devoted to the proof of Lemma 4.1. Let us recall that $q^*(l) = m(m^l - 1)/(m - 1)$. We first prove

Lemma B.1. *Let p be an integer and let $(A(I); |I| \le l)$ be elements of R such that, in R,*

$$\sum_{|I| \le l} A(I)C_r(I) = 0 \qquad \textit{for all } r \textit{ in } [0, q^*(l) + p]; \tag{B.1}$$

then

$$\sum_{|I| \le l-1} A(I * k)C_r(I) = 0 \qquad \textit{for all} \quad r \in [0, p] \quad \textit{and for all} \quad k \in [1, m]. \tag{B.2}$$

We prove Lemma B.1 by induction on p.

Step 1. We prove Lemma B.1 for $p = 0$. The proof is similar to the proof of [G, Lemma 1, p. 158]. Note that for $p = 0$, (B.2) becomes

$$A(i) = 0 \qquad \text{for all } i \text{ in } [1, m]. \tag{B.3}$$

Assume that, for example

$$A(1) \ne 0. \tag{B.4}$$

Then, by induction on l, we easily check, using (4.16), (4.17), (4.18), and (B.1), that

for all r in $[0, q^*(l) - 1]$, $b_1^{(r)}$ is a polynomial in $A(I)/A(1)$, and $b_i^{(j)}$ where $0 < |I| \le l$, $i \ne 1$, and $j \le r$. (B.5)

But the cardinality of $\{I; 0 < |I| \le l \text{ and } I \ne 1\}$ is $q^*(l) - 1$; hence (B.5) cannot be true.

Step 2. Assuming that Lemma B.1 is true for p, we prove now that it is also true for $p + 1$. Let $(A(I); |I| \le 1)$ be elements in R such that

$$\sum_{|I| \le l} A(I)C_r(I) = 0 \qquad \text{for all } r \text{ in } [1, q^*(l) + p + 1]. \tag{B.6}$$

From (4.15) we obtain for all k in $[1, m]$

$$\sum_{|I| \le l-1} A(I * k)C_{p+1}(I) = \sum_{i=1}^{m} \sum_{|J| \le l-2} b_i A(i * J * k)C_p(J) + \sum_{|I| \le l-1} A(I * k)C_p(I). \tag{B.7}$$

Since Lemma B.1 is true for p, differentiating (in the differential field R defined by $\dot{b}_i^{(j-1)} = b_i^{(j)}$) (B.2) for $r = p$, we have, for all k in $[1, m]$,

$$\sum_{|I| \le l-1} A(I * k)\dot{C}_p(I) = -\sum_{|I| \le l-1} \dot{A}(I * k)C_p(I). \tag{B.8}$$

Next using (B.6) and (4.15) we have, for all r in $[0, q^*(l) + p]$,

$$-\sum_{|I|\le l} \dot{A}(I)C_r(I) + \sum_{i=1}^{m} \sum_{|J|\le l-1} b_i A(i * J)C_r(J) = 0. \tag{B.9}$$

Hence by Lemma B.1 for p, we get, for all k in $[1, m]$,

$$\sum_{|I|\le l-1} \dot{A}(I * k)C_p(I) + \sum_{i=1}^{m} \sum_{|J|\le l-2} b_i A(i * J * k)C_p(J) = 0. \tag{B.10}$$

Finally, from (B.7), (B.8), and (B.10) we get

$$\sum_{|I|\le l-1} A(I * k)C_{p+1}(I) = 0 \qquad \text{for all } k \text{ in } [1, m]. \tag{B.11}$$

This completes the proof of Lemma B.1. ■

We now deduce Lemma 4.1 from Lemma B.1.

Let $(A(I); 0 < |I| \le l)$ be elements of $\mathbb{R}$ such that

$$\sum_{0<|I|\le l} A(I)C_r(I) = 0 \qquad \text{for all } r \text{ in } [1, q(l)]. \tag{B.12}$$

We take $A(\emptyset) = 0$. From (B.12) and Lemma B.1 with $p = q(l) - q^*(l) = q(l-1)$ we have

$$\sum_{|I|\le l-1} A(I * k)C_r(I) = 0 \qquad \text{for all} \quad r \in [0, q(l-1)] \qquad \text{and for all} \quad k \in [1, m]. \tag{B.13}$$

In particular,

$$A(k) = 0 \qquad \text{for all } k \text{ in } [1, m] \tag{B.14}$$

and, still for all k in $[1, m]$,

$$\sum_{0<|I|\le l-1} A(I * k)C_r(I) = 0 \qquad \text{for all } r \text{ in } [0, q(l-1)]. \tag{B.15}$$

An easy induction argument on l gives Lemma 4.1. ■

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