SLIDING MOTION IN FILIPPOV DIFFERENTIAL SYSTEMS: THEORETICAL RESULTS AND A COMPUTATIONAL APPROACH

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Abstract. In this work, we discuss some theoretical and numerical aspects of solving differential equations with discontinuous right-hand sides of Filippov type. In particular: (i) we propose second order corrections to the theory of Filippov, (ii) we provide a systematic and non-ambiguous way to define the vector field on the intersection of several surfaces of discontinuity, and (iii) we propose, and implement, a numerical method to approximate a trajectory of systems with discontinuous right-hand sides, and illustrate its performance on a few examples.

1. Introduction and Motivation

In this work, we discuss some theoretical and numerical questions related to differential equations with discontinuous right-hand-side. The setting we consider is the classical one of Filippov, see [16], whereby one seeks a solution of an initial value problem of ordinary differential equations in which the right-hand side (the vector field) varies discontinuously as the solution trajectory reaches one or more surfaces, called *discontinuity or switching surfaces*, but it is otherwise smooth; in the literature, these are called "piecewise smooth systems", hereafter PSW for short. The values where a trajectory reaches a discontinuity surface are called *events*, and we will henceforth assume that the events are isolated. In general, there are a number of possible outcomes as the solution reaches a discontinuity surface. For example, in so-called *impact* systems, the solution experiences a jump discontinuity; see [36]. However, in this work, we will only consider the case where the solution remains continuous (though not necessarily differentiable) past an event point. In this case, loosely speaking, there are two things which can occur as we reach a surface of discontinuity: we may cross it, or we may stay on it, in which case a description of the motion on the surface will be required, *sliding motion*. This latter case is particularly interesting and important, and calls for a separate theoretical and numerical analysis.

Systems with discontinuous right-hand sides appear pervasively in applications of various nature (see, e.g. [5, 8, 18, 19, 23, 24, 30, 33]). For a sample of references in the context of control, see e.g. [38, 39, 37], and in the context of biological systems, see e.g. [7, 10, 11, 19, 33]; for works on the class of complementarity systems, see [22], for works from the point of view of bifurcations of dynamical-systems see [13, 26, 25, 28, 29]; and, of course, see the classical references [4, 16, 38, 39] for a thorough theoretical introduction to these systems.

Because of their ubiquity in applications of biological and engineering nature, PWS systems are receiving a lot of attention. To witness, we mention the recent books [1, 12] which deal with specific questions of bifurcations and simulations for PWS systems. Indeed, many studies on PWS-systems rely on simulation, and the cited text [1] has a nice collection of different case studies for which specific numerical techniques have been devised. For completeness, we also refer to the

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works [14, 15, 17, 31, 32, 34] for a representative sample of the kind of questions which have been investigated by the numerical analysis community.

In spite of the attention that they have been receiving, systems with discontinuous right-hand sides still present several outstanding theoretical and practical challenges. In particular, the widely adopted Filippov extension to define the vector field in a sliding regime is not properly defined when sliding occurs on the intersection of two (or more) surfaces. This difficulty can be avoided for a special class of PWS-systems following the approach of Stewart, which uses a construction based on linear complementarity problems (see [35]), but for general systems of the type we consider in this work Stewart's technique is not directly applicable, and the techniques which have been used in practice are: (i) *smoothing* out the vector field, see [9, 33]; (ii) *blending*, i.e., essentially interpolating, the vector fields in the neighborood of the discontinuity surface(s), see [2, 3, 12]; (iii) modify further the vector field with a so-called transition phase (*switch model*) which connects in a gentle way the behavior in the regions around and on the discontinuity surface; see [27, 28]. We remark that these techniques alter the nature of the original PWS-system in either a global or local fashion. Another difficulty, with the Filippov extension of the vector field in the sliding regime, arises when the first order conditions on which the Filippov extension is based are violated. In fact, precisely these two points, sliding on intersection of several surfaces and violation of the first order Filippov theory, have motivated our work, whose main goals are: to reinterpret the 1st order theory of Filippov and to propose second order corrections to the theory; to provide a systematic, and non-ambiguous, way to define the vector field on the intersection of several surfaces of discontinuity; and, finally, to consider a numerical method to approximate a trajectory of a PWS-system. A main feature of our numerical approach is its ability to reach the sliding surface(s) from one side only.

2. Background on Filippov Theory

Here we review the basics of the theory of Filippov about solutions of PWS-systems (see [4, 16, 38, 39]).

We begin by describing a PWS-system in its simplest modelization, the case in which a (global) hypersurface partitions the state space in two regions. More complicated situations arise as generalizations of this simple model. Consider the nonlinear system with discontinuous right-hand side:

$$x'(t) = f(x(t)) = \begin{cases} f_1(x(t)) \,, & x \in S_1 \,, \\ f_2(x(t)) \,, & x \in S_2 \,, \end{cases} \qquad x(0) = x_0 \in \mathbb{R}^n \,. \tag{2.1}$$

The state space $\mathbb{R}^n$ is split into two subspaces S_1 and S_2 by a hypersurface Σ such that $\mathbb{R}^n = S_1 \cup \Sigma \cup S_2$. The hypersurface is defined by a scalar *indicator* (or *event*) function $h : \mathbb{R}^n \to \mathbb{R}$, so that the subspaces S_1 and S_2, and the hypersurface Σ, are implicitly characterized as

$$\Sigma = \{x \in \mathbb{R}^n |\ h(x) = 0\} \,, \quad S_1 = \{x \in \mathbb{R}^n |\ h(x) < 0\} \,, \quad S_2 = \{x \in \mathbb{R}^n |\ h(x) > 0\} \,. \tag{2.2}$$

We will assume that $h \in \mathcal{C}^k$, $k \geq 2$, and that $\nabla h(x) \neq 0$ for all $x \in \Sigma$. Thus, the unit normal $\mathtt{n}$ to Σ, perpendicular to the tangent plane $T_x(\Sigma)$ at $x \in \Sigma$, is given by

$$\mathtt{n}(x) = \frac{\nabla(h(x))}{\|\nabla(h(x))\|_2} \,, \quad \|\mathtt{n}(x)\| = 1, \ \ \forall x \in \Sigma \,. \tag{2.3}$$

In (2.1), the right-hand side $f(x)$ can be assumed to be smooth on S_1 and S_2 separately, but it is usually discontinuous across Σ. To be precise, we will assume that f_1 is $C^k, k \geq 1$, on $S_1 \cup \Sigma$ and f_2 is $C^k, k \geq 1$, on $S_2 \cup \Sigma$, but we will not assume that f_1, respectively f_2, extends smoothly also in S_2, respectively S_1. Often, it is assumed explicitly or implicitly that f_1 and f_2 are defined smoothly everywhere; e.g., see [16, 29, 38]. However, this seem a strong restriction to us, and we prefer not even to assume that f_1, respectively f_2, are defined in S_2, respectively S_1.

Obviously, in (2.1), $f(x(t))$ is not defined if $x(t)$ is on Σ. We stress that this means exactly that: $f(x(t))$ is not defined from (2.1) when $x(t)$ is on Σ. In the model (2.1), there is freedom on how to extend the vector field on Σ, and the way that this freedom is resolved must ultimately be weighted against our ability to model situations of practical interest, and to have mathematical backing for existence of solutions. A most important and widely accepted way to resolve this freedom is to consider the set valued extension $F(x)$ below:

$$x'(t) \in F(x(t)) = \begin{cases} f_1(x(t)) & x \in S_1 \\ \overline{\text{co}}\,\{f_1(x(t), f_2(x(t)\} & x \in \Sigma \\ f_2(x(t)) & x \in S_2 \end{cases} , \tag{2.4}$$

where $\overline{\text{co}}(A)$ denotes the smallest closed convex set containing A. In our particular case:

$$\overline{\text{co}}\,\{f_1, f_2\} = \{f_{_F} \in \mathbb{R}^n \ : \ f_{_F} = (1-\alpha)f_1 + \alpha f_2,\ \alpha \in [0,1]\} \ . \tag{2.5}$$

The extension (or convexification) of a discontinuous system (2.1) into a convex differential inclusion (2.4) is known as *Filippov convex method*. Existence of solutions of (2.4) can be guaranteed with the notion of upper semi-continuity of set-valued functions ([4], [16]).

Definition 2.1 (Solution in the sense of Filippov; [16])**.** An absolutely continuous function $x : [0,\tau) \to \mathbb{R}^n$ is said to be a solution of (2.1) in the sense of Filippov, if for almost all $t \in [0,\tau)$ it holds that

$$x'(t) \in F(x(t)) \ ,$$

where $F(x(t))$ is the closed convex hull in (2.5).

In this work, we are only going to consider the case of solutions which are continuous, though not necessarily differentiable. This greatly simplifies the development of numerical methods for approximating the relevant solution.

Now, consider a trajectory of (2.1), and suppose that $x_0 \notin \Sigma$, and thus, without loss of generality, we can think that $x_0 \in S_1$. The interesting case is when there exists a finite time at which the solution reaches Σ. At this point, one must decide what happens next. Loosely speaking, there are two possibilities: (a) we exit Σ and enter into S_1 or S_2; (b) we remain in Σ with a yet to be defined vector field. Filippov deviced a very powerful theory which helps to decide what to do in this situation, and how to define the vector field in case (b). We summarize it below.

2.1. **Filippov first order theory.** Let $x \in \Sigma$ and let $\mathtt{n}(x)$ be the normal to Σ at x. Let $\mathtt{n}^T(x)f_1(x)$ and $\mathtt{n}^T(x)f_2(x)$ be the projections of $f_1(x)$ and $f_2(x)$ onto the normal to the hypersurface Σ.

(a) *Transversal Intersection.* In case in which, at $x \in \Sigma$, we have

$$[\mathtt{n}^T(x)f_1(x)] \cdot [\mathtt{n}^T(x)f_2(x)] > 0 \ , \tag{2.6}$$

then we will leave Σ. We will enter S_1, when $\mathtt{n}^T(x)f_1(x) < 0$, and will enter S_2, when $\mathtt{n}^T(x)f_1(x) > 0$. In the former case we will have (2.1) with $f = f_1$, in the latter case with $f = f_2$. Any solution of (2.1) with initial condition not in Σ, reaching Σ at a time t_1, and having a transversal intersection there, exists and is unique.

(b) *Sliding Mode.* In case in which, at $x \in \Sigma$, we have

$$[\mathtt{n}^T(x)f_1(x)] \cdot [\mathtt{n}^T(x)f_2(x)] < 0 \ , \tag{2.7}$$

then we have a so-called sliding mode through x. This is further classified as *attracting*, or *repulsive*, depending on the following situation.

(b-1) *Attracting Sliding Mode.* An attracting sliding mode at Σ occurs if

$$[\mathtt{n}^T(x)f_1(x)] > 0 \quad \text{and} \quad [\mathtt{n}^T(x)f_2(x)] < 0, \qquad x \in \Sigma\ , \tag{2.8}$$

where the inequality signs depend of course on the definition of $S_{1,2}$ in (2.2). When we have (2.8) satisfied at $x_0 \in \Sigma$, a solution trajectory which reaches x_0 does not leave Σ, and will therefore have to move along Σ. Filippov's theory provides an extension to the vector field on Σ, consistent with the interpretation in (2.5), giving rise to *sliding motion*. During the sliding motion the solution will continue along Σ with time derivative $f_{_F}$ given by

$$f_{_F}(x) = (1-\alpha(x))f_1(x) + \alpha(x)f_2(x)\,. \tag{2.9}$$

Here, $\alpha(x)$ is the value for which $f_{_F}(x)$ lies in the tangent plane T_x of $h(x)$ at x, that is the value for which $\mathtt{n}^T(x)f_{_F}(x) = 0$. This gives

$$\alpha(x) = \frac{\mathtt{n}^T(x)f_1(x)}{\mathtt{n}^T(x)(f_1(x)-f_2(x))}. \tag{2.10}$$

Observe that a solution having an attracting sliding mode exists and is unique, in forward time.

(b-2) *Repulsive Sliding Mode.* If

$$[\mathtt{n}^T(x)f_1(x)] < 0 \quad \text{and} \quad [\mathtt{n}^T(x)f_2(x)] > 0, \qquad x \in \Sigma\ , \tag{2.11}$$

we have a *repulsive sliding mode*. Repulsive sliding modes do not lead to uniqueness (at any instant of time one may leave with f_1 or f_2), and we will not further consider repulsive sliding motion in this work.

All cases not summarized above require a more careful analysis, since they are not covered by the 1st order theory presented. The next simplest situation is that in which we had been having attracting sliding motion, but for some value of t, at the point $x = x(t)$ (2.8) is no longer satisfied. Consistently with the nature of Filippov's theory, one should assume that one of these two things has occurred (but not both):

$$[\mathtt{n}^T(x)f_1(x)] = 0 \quad \text{or} \quad [\mathtt{n}^T(x)f_2(x)] = 0\ . \tag{2.12}$$

In the former case, see (2.10), one will have $\alpha = 0$ and $f_{_F} = f_1$, in the latter case one will have $\alpha = 1$ and $f_{_F} = f_2$. In other words, one of the two original vector fields is already in the tangent plane T_x. We expect that in the case of $\alpha = 0$ we will enter in S_1 and in case $\alpha = 1$ we will enter in S_2. To verify whether or not –and when– this will occur, or if one has sliding motion on Σ, will be discussed next.

3. Higher order conditions

In case of (2.12), we need to look at derivative terms to determine if we should leave Σ or if sliding motion should (continue to) take place.

Let us introduce some notation, and do some elementary geometrical considerations. For all $x \in \Sigma$, let

$$g_1(x) = (\nabla h(x))^T f_1(x)\ , \quad g_2(x) = (\nabla h(x))^T f_2(x)\ , \quad g(x) = g_1(x)g_2(x)\ . \tag{3.1}$$

Consider the sets

$$\Sigma_{_T} = \{x \in \Sigma\ :\ g(x) > 0\}\,, \quad \text{and} \quad \Sigma_{_S} = \{x \in \Sigma\ :\ g(x) < 0\}\,,$$

which we will call the sets of *transversality* and of *sliding points*, respectively. Clearly, $\Sigma_{_T}$ and $\Sigma_{_S}$ are open and disjoint (possibly, empty). Define also the following *exit sets*:

$$E_1 := \{x \in \Sigma\ :\ g_1(x) = 0\}\ ,\ E_2 := \{x \in \Sigma\ :\ g_2(x) = 0\}\ ,$$

$$E := E_1 \cup E_2 = \overline{\Sigma}_{T} \cap \overline{\Sigma}_{S} = \{x \in \Sigma \ : \ g(x) = 0\} \ .$$

In general, a point $x \in \Sigma$ lies on an $(n-1)$-dimensional manifold of points in Σ (since $\nabla h(x) \neq 0$). Likewise, if a point x belongs to E_1, or E_2, in general it will belong to an $(n-2)$-dimensional manifold of such points in Σ; e.g., this is the case if the vector $\nabla g_1(x)$, respectively $\nabla g_2(x)$, is not parallel to the vector $\nabla h(x)$, which is to be generically expected for smooth $h(x)$ and $f_i(x)$, $i = 1, 2$. As a consequence, we should expect that a smooth curve (i.e., a trajectory) in Σ may encounter E_1 and/or E_2, though not $E_1 \cap E_2$, whereas a curve in E_1 or E_2 may encounter $E_1 \cap E_2$.

So, suppose we are having a trajectory on Σ, $x(t)$, which at some instant of time (say, $t = 0$) reaches E. At this point, we must decide if the trajectory will leave Σ (and with which vector field), or if it will stay on Σ, in which case we need to define the vector field for sliding motion on Σ. One feature which we want to preserve is that the solution –regardless of whether it stays on Σ or leaves it– will start with the value $x(0)$. The idea is to look at left (and right) limit expansions of the functions $g_1(x(t))$ and $g_2(x(t))$ and enforce smoothness of the solution.

In what follows we will freely assume that f_1, f_2 and h are sufficiently differentiable in their respective domains of definitions, so that all derivatives we take make sense.

With the agreement that $t = 0^-$ really means $\lim_{t\to 0^-}$, for t in a left neighborhood of $t = 0$, we have:

$$g_i(x(t)) = g_i(x(0)) + t\left[\frac{\partial}{\partial x} g_i(x(t))x'(t)\right]_{t=0^-} + \frac{t^2}{2}\left[(x'(t))^T \frac{\partial^2}{\partial x^2} g_i(x(t))x'(t) + \frac{\partial}{\partial x} g_1(x(t))x''(t)\right]_{t=0^-} + O(t^3)\,,$$

where $\frac{\partial^2}{\partial x^2} g_i$ is the Hessian of g_i, for $i = 1, 2$. Write this expression compactly as

$$g_i(x(t)) = A_i + tB_i^- + \frac{t^2}{2}C_i^- + O(t^3)\,, \tag{3.2}$$

where (for $i = 1, 2$)

$$A_i = g_i(x(0))\ , \quad B_i^- = \left[\frac{\partial}{\partial x} g_i(x(t))x'(t)\right]_{t=0^-}\ , \qquad \text{and}$$

$$C_i^- = \left[(x'(t))^T \frac{\partial^2}{\partial x^2} g_i(x(t))x'(t) + \frac{\partial}{\partial x} g_i(x(t))x''(t)\right]_{t=0^-}\ .$$

In the above, the quantities $x'(0^-), x''(0^-)$ are not ambiguous, since the solution for $t \to 0^-$ is well defined. With abuse of notation, we also let $t = 0^+$ in lieu of $\lim_{t\to 0^+}$, and for t in a right neighborhood of $t = 0$ set:

$$g_i(x(t)) = A_i + tB_i^+ + \frac{t^2}{2}C_i^+ + O(t^3)\,, \tag{3.3}$$

where

$$B_i^+ = \left[\frac{\partial}{\partial x} g_i(x(t))x'(t)\right]_{t=0^+}\ , \quad C_i^+ = \left[(x'(t))^T \frac{\partial^2}{\partial x^2} g_i(x(t))x'(t) + \frac{\partial}{\partial x} g_i(x(t))x''(t)\right]_{t=0^+}\ ,$$

for $i = 1, 2$. Now, of course, the values B_i^+, C_i^+ (that is, the expressions $x'(0^+), x''(0^+)$) depend on how we will define the solution past the value $x(0)$.

We summarize several possibilities of motion which can be observed.

(I) $A_1 \neq 0$, $A_2 \neq 0$. This is the basic case covered by Filippov's theory.
- $A_1 > 0$, $A_2 > 0$. We leave Σ to enter in S_2 with vector field $f_2(x(0))$.
- $A_1 < 0$, $A_2 < 0$. We leave Σ to enter in S_1 with vector field $f_1(x(0))$.
- $A_1 > 0$, $A_2 < 0$. This is the case of attractive sliding motion. We stay on Σ with $f_{F}(x(0)) = (1-\alpha)f_1(x(0)) + \alpha f_2(x(0))$, and $\alpha = \frac{A_1}{A_1 - A_2}$.

- $A_1 < 0$, $A_2 > 0$. This is the ill posed case of repulsive sliding motion, which we do not consider.

(II) $A_1 = 0$, $A_2 \neq 0$. We should expect this case to arise when coming from attractive sliding motion on Σ, or –less likely– when coming from motion outside Σ.

- Coming from attractive sliding on Σ, and $A_2 < 0$. Since $g_1(x(t)) > 0$ for $t < 0$, from (3.2) we must have $B_1^- \leq 0$. Since $A_1 = 0$, then $x'(0^-) = f_1(x(0))$, which we assume to be nonzero.
  - Let $B_1^- < 0$ and we define $x'(0^+) = f_1(x(0))$. Thus, $x(t)$ extends differentiably past $t = 0$, $g_1(x(t)) < 0$ for $t > 0$ (small) and the motion leaves Σ and enters in S_1 with $f_F = f_1$. We call this the case of a *1st order exit condition*.
  - If $B_1^- = 0$, then must have $C_1^- \geq 0$. Assume $C_1^- > 0$. Then, define $x'(0^+) = x'(0^-) = f_1(x(0))$ and $x''(0^+) = x''(0^-)$ so that $B_1^+ = B_1^-$, $C_1^+ = C_1^-$, and $x(t)$ extends past $t = 0$ as a $\mathcal{C}^2$ function. So, $g_1(x(t)) > 0$ for $t > 0$ (small) and sliding motion continues with f_1. Notice that $f_F = f_1$, which lies on the tangent plane to Σ at $x(0)$, and we also have
  $$\nabla g_1(x) \;=\; f_1^T(x) h_{xx}(x) + \nabla h(x)^T D f_1(x) \ ,$$
  and thus $(\nabla g_1(x))^T f_1(x) = 0$, which implies that f_1 is also tangent to E_1. In other words, in this situation, we have sliding motion on the lower-dimensional submanifold E_1 of Σ. This motion will continue until exit conditions –relatively to this sliding motion– will be met, or of course a fixed point will be reached. It is worth remarking that the solution in both cases just described exists, is unique, and satisfies (locally) a smooth differential system. [If $f_1(x(0)) = 0$ then $x(0)$ becomes an equilibrium.]
- Coming from S_1.
  - $A_2 < 0$. Then, the previous analysis applies (because in the expansion of g_1 we need to use $x'(0^-) = f_1(x(0))$ and $g_1(x(t)) < 0$ for $t < 0$ small).
  - $A_2 > 0$. If $B_1^- < 0$, we have an ill-posed problem with repulsive sliding. If $B_1^- = 0$ and $C_1^- > 0$, then we have a crossing into S_2, with $x'(0^+) = f_2(x(0))$ and there is uniqueness in forward time.
- Coming from S_2. Then $A_2 < 0$, $x'(0^-) = f_2(x(0))$, and $x'(0^+) = f_1(x(0))$.
  - $B_1^+ < 0$. Then, there is crossing into S_1 with $x'(0^+) = f_1(x(0))$.
  - $B_1^+ > 0$. Then, there is sliding on Σ with $f_F(x(0)) = f_1(x(0))$.

(III) $A_1 \neq 0$, $A_2 = 0$. This is the same as case (II) (change the indices 1 and 2).

(IV) $A_1 = A_2 = 0$. We are on E, and we should expect this (unlikely) situation to arise while sliding along E_1 or E_2. This case can lead to very complicated situations, and we do not fully understand all possibilities; nevertheless, there are some situations in which we can decide whether we should expect to leave Σ or to slide on it. For example, if $B_1^- = B_2^- = 0$, then we can have a well posed problem before and after $t = 0$, with either crossing or sliding. The latter may be realized with $C_1^- = C_1^+ > 0$ and $C_2^- = C_2^+ < 0$, with $\alpha = C_1/(C_1 - C_2)$.

Because of its importance, we summarize the situation of first order exit conditions, as a definition.

Definition 3.1 (1st order Exit Conditions)**.** Using the notation of (3.2), we say that a sliding trajectory on Σ will leave Σ by fullfilling first order exit conditions when either case (a) or (b) below are satisfied:

(a) $\quad A_1 = 0, A_2 < 0, B_1^- < 0$, $\qquad$ (b) $\quad A_1 > 0, A_2 = 0, B_2^- > 0$.

We notice that, for smooth functions h, f_1, f_2, generically we expect a solution to exit Σ precisely because of satisfying 1st order exit conditions. And, clearly, in case (a) of Definition 3.1 we will enter in S_1 with vector field f_1, while in case (b) we will enter S_2 with f_2. At times, however, special structure in the problem makes it necessary to deal with higher order conditions.

Example 3.2 (Control Problem; see [23])**.** As concrete illustration of the case (IV) above, $A_1 = A_2 = 0$, we consider a model from the control literature. The problem is

$$\begin{aligned} x'(t) &= Ax(t) + bu(t) \\ y &= c^T x \end{aligned} \tag{3.4}$$

with

$$u(t) = -\text{sign}(y(t)) = \begin{cases} 1 & y(t) < 0 \\ [-1,1] & y(t) = 0 \\ -1 & y(t) > 0 \end{cases}$$

and where the state variable $x \in \mathbb{R}^n$, and $A \in \mathbb{R}^{n\times n}$ and $b, c \in \mathbb{R}^n$. In our notation, we have $h(x) = c^T x$, $S_1 = \{x \in \mathbb{R}^n |\ h(x) < 0\}$, $S_2 = \{x \in \mathbb{R}^n |\ h(x) > 0\}$, $\Sigma = \{x \in \mathbb{R}^n |\ c^T x = 0\}$, and $\nabla h(x) = c$ for all $x \in \Sigma$. In particular, we have the fields f_1, f_2 in S_1 and S_2, and the Filippov vector field $f_F(x) = (1-\alpha)f_1 + \alpha f_2$ during sliding motion in Σ, given by:

$$\begin{aligned} f_1(x) &= Ax + b \\ f_F(x) &= Ax + (1-2\alpha)b\,. \\ f_2(x) &= Ax - b \end{aligned} \tag{3.5}$$

In this case, for the functions $g_i, i = 1, 2$, we have

$$g_1(x) = c^T Ax + c^T b\ , \quad g_2(x) = c^T Ax - c^T b\ ,$$

and we observe that $\nabla g_1 = \nabla g_2 = A^T c$ and $\frac{\partial^2}{\partial x^2} g_1 = \frac{\partial^2}{\partial x^2} g_2 = 0$.

Suppose that $x_0 \in \Sigma$. We have to consider two cases: $c^T b \neq 0$ and $c^T b = 0$. In the first case, it is simple to realize that if $|c^T Ax_0/c^T b| < 1$ then attracting sliding motion takes place. The case where the data are such that $c^T b = 0$ is considerably more complicated, as we see below.

With our previous notation, we have $A_1 = c^T Ax_0 + c^T b$ and $A_2 = c^T Ax_0 - c^T b$. Then, we will have $A_1 = A_2$ and thus cannot have sliding motion unless $A_1 = A_2 = 0$, that is $c^T Ax_0 = 0$ (these are called sliding conditions of order 2 in [23]). Furthermore, since $c^T b = 0$, we cannot have reached x_0 coming from a 1st order attracting sliding motion on Σ. Thus, to decide how to proceed from x_0 we must look at g_1 and g_2 near Σ, for $t < 0$. We have (recall (3.5) for how the vector fields are defined below/above Σ):

$$t \le 0: \quad g_1(x(t)) = c^T Ax_0 + t(c^T Ax'(0^-)) + \frac{t^2}{2}(c^T Ax''(0^-)) + \cdots\ ,$$

and here we have $x'(0^-) = Ax_0 + b$ so that $B_1^- = c^T A^2 x_0 + c^T Ab$ and we will need $B_1^- < 0$ (so to have $g_1(x(t)) > 0$ for $t < 0$). Likewise,

$$t \le 0: \quad g_2(x(t)) = c^T Ax_0 + tB_2^- + \cdots\ , \quad B_2^- = c^T A^2 x_0 - c^T Ab\ ,$$

and we need $B_2^- > 0$ (so to have $g_2 < 0$ for $t < 0$). In other words, we will need to have $c^T Ab < 0$ and $|c^T A^2 x_0| < |c^T Ab|$. This would give us the choice for α in the Filippov vector field: $\alpha = 1/2 + c^T A^2 x/(2c^T Ab)$. Now, if we were to have attractive sliding motion on Σ past $t = 0$, we would need to have $g_1(x(t)) > 0$ and $g_2(x(t)) < 0$ for $t > 0$; however, this is not possible, since (for $t > 0$) $g_1(x(t)) = g_2(x(t))$, so one cannot have attractive sliding motion in this case. The above reasoning can be continued, by considering data such that $c^T Ab = 0$, and so forth. We summarize the general case in Table 1, where the following convention has been adopted. Under the heading "Data" we report on constraints on the data of the problem, under the heading "order" we report

Table 1. Control Problem Example 3.2

Data Constraints	Sliding Set	Order	α	Restrictions
$c^Tb \neq 0$	$c^Tx = 0$	1	$\frac{1}{2}(1+\frac{c^TAx}{c^Tb})$	$\left\lvert\frac{c^TAx}{c^Tb}\right\rvert < 1$
$c^Tb = 0$	$c^TAx = 0$	2	$\frac{1}{2}(1+\frac{c^TA^2x}{c^TAb})$	$\left\lvert\frac{c^TA^2x}{c^TAb}\right\rvert < 1$
$c^TAb = 0$	$c^TA^2x = 0$	3	$\frac{1}{2}(1+\frac{c^TA^3x}{c^TA^2b})$	$\left\lvert\frac{c^TA^3x}{c^TA^2b}\right\rvert < 1$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$c^TA^{r-1}b = 0$	$c^TA^rx = 0$	$r+1$	$\frac{1}{2}(1+\frac{c^TA^{r+1}x}{c^TA^rb})$	$\left\lvert\frac{c^TA^{r+1}x}{c^TA^rb}\right\rvert < 1$

the order of the sliding sets as defined in [23]. It has to be understood that having the k-th order sliding set implies at once all the previous data constraints and sliding sets. In other words, for k-th order sliding set one really has the data constraints: $c^TA^jb = 0$, $j = 0, \dots, k-2$, and the k-th order sliding set is the intersection of all hyperplanes $c^TA^jx = 0$, $j = 0, \dots, k-1$. Finally, in the column α we report the necessary value for α we must have in the Filippov vector field if we had to have sliding motion (though we reiterate that this motion cannot be attractive), and in the column "Restrictions" are stated the restriction needed in order to have a feasible α.

As we said, in case in which $c^Tb = 0$, there cannot be attractive sliding motion on the 2nd order sliding set $\{x \in \mathbb{R}^n \ : \ c^Tx = c^TAx = 0\}$. In this case, what may happen is that the trajectory can wind around this 2nd order sliding set, without actually being on it. This kind of phenomenon is known as *chattering* (see [23]) and we will see an instance of it in the numerical experiments; see Section 6.

4. Projections, tangent vectors, and two and more sliding surfaces

In this section we revisit Filippov construction. We are motivated to do this because of the existing ambiguity on how to define the vector field in case one has sliding motion on the intersection of two, or more, surfaces; see Section 4.2. To arrive at an appropriate way to define the vector field in this sliding regime, we need to appreciate the key points which make Filippov's theory so powerful: (i) It gives a constructive, and simple, way to define the vector field for sliding motion on one surface, (ii) It is possible to obtain existence (and uniqueness) results because of the theory of convex differential inclusions, (iii) It is a theory which automatically accounts not only for sliding motion, but also for first order exit conditions, that is for when we need to leave the sliding surface. In our reinterpretation, we will try to retain these key features, while providing a systematic way to define the vector fields in the different regimes.

4.1. Projections and tangent vectors.

Let us consider the case of the PWS-system (2.1), and specifically the situation in which we anticipate there being attractive sliding motion on Σ, see (2.7). Filippov's construction rests on using (2.9) and then finding α (which is of course a function of $x \in \Sigma$) so that $f_F \in T_x(\Sigma)$, the tangent plane at x to Σ. We start from this last consideration.

Let $\pi_\Sigma(x)$ be the orthogonal projection onto $T_x(\Sigma)$ that is $\pi_\Sigma(x) = I - \mathtt{n}(x)\mathtt{n}^T(x)$. Because of (2.3), this projection varies smoothly in $x \in \Sigma$. We consider as possible sliding vector field a vector field, call it f_S, which is a convex linear combination of $\pi_\Sigma f_1$ and $\pi_\Sigma f_2$:

$$f_S(x) = (1-\beta(x))\pi_\Sigma(x)f_1(x) + \beta(x)\pi_\Sigma(x)f_2(x), \quad \text{with } 0 \le \beta \le 1 \ , \tag{4.1}$$

where $\beta(x)$ is required to depend smoothly on $x \in \Sigma$.

From (4.1) it follows that:

$$f_S(x) = (1-\beta(x))f_1(x) + \beta(x)f_2(x) - (1-\beta(x))\mathtt{n}(x)\mathtt{n}^T(x)f_1(x) - \beta(x)\mathtt{n}(x)\mathtt{n}^T(x)f_2 \ ,$$

and recalling the form of $\alpha(x)$ in (2.10) in the expression for $f_F(x)$ in (2.9), it easily follows that $f_S(x) = f_F(x)$, if $\beta(x) = \alpha(x)$. Obviously, in one dimension (that is when Σ is a curve in the plane, $n = 2$), f_S and f_F are both in the direction of the unique tangent vector to the curve.

Here we propose to take $\beta(x)$ as a first order rational function:

$$\beta(x) = \frac{a_1 \mathtt{n}^T(x) f_1(x) + b_1 \mathtt{n}^T(x) f_2(x) + c_1}{a_2 \mathtt{n}^T(x) f_1(x) + b_2 \mathtt{n}^T(x) f_2(x) + c_2} \tag{4.2}$$

with a_1, b_1, c_1 and a_2, b_2, c_2 real coefficients. To find these coefficients (which naturally must depend smoothly on $x \in \Sigma$), we impose the interpolation (exit) conditions:

$$\begin{cases} \beta(x) = 0 & \text{when} \quad \mathtt{n}^T(x) f_1(x) = 0, \\ \beta(x) = 1 & \text{when} \quad \mathtt{n}^T(x) f_2(x) = 0, \end{cases}$$

that is $\beta(x)$ must have the following form:

$$\beta(x) = \frac{\mathtt{n}^T(x) f_1(x)}{\mathtt{n}^T(x)(f_1(x) - a f_2(x))} \ , \tag{4.3}$$

with $a_1 = a_2$, $a = -b_2/a_1$, $b_1 = c_1 = c_2 = 0$ and where $a = a(x) > 0$ depends smoothly on $x \in \Sigma$. Since the sliding condition (2.7) holds, that is $\mathtt{n}^T(x) f_1(x) > 0$ and $\mathtt{n}^T(x) f_2(x) < 0$, (4.3) gives values for $\beta(x)$ between 0 and 1.

A simple verification shows that taking $a = 1$ in (4.3) gives the Filippov vector field with $\beta(x) = \alpha(x)$. As a consequence of the above construction, we conclude that Filippov choice for the vector field, $f_F(x)$, in the attractive sliding mode regime, may be viewed as having chosen the vector field according to (4.1) and then having imposed first order exit conditions in the rational form (4.2), and fixing $a = 1$ in (4.3).

4.2. **Two and more sliding surfaces.** Let us now consider the case in which we have two different surfaces Σ_1 and Σ_2, characterized as zero sets of functions $h_1(x)$ and $h_2(x)$:

$$\Sigma_1 \ := \ \{x \in \mathbb{R}^n \ : \ h_1(x) = 0\}\,, \quad \Sigma_2 \ := \ \{x \in \mathbb{R}^n \ : \ h_2(x) = 0\}\,. \tag{4.4}$$

Similarly to the assumptions of Section 2, the functions $h_1(x)$ and $h_2(x)$ are assumed to be $\mathcal{C}^k$ functions ($k \geq 2$) and moreover $\nabla h_1(x) \neq 0$, for all $x \in \Sigma_1$, and $\nabla h_2(x) \neq 0$, for all $x \in \Sigma_2$. So, we have well defined unit normals $\mathtt{n}_1(x)$ and $\mathtt{n}_2(x)$ to $T_x(\Sigma_1)$ and $T_x(\Sigma_2)$, respectively. We will henceforth assume that $\mathtt{n}_1(x)$ and $\mathtt{n}_2(x)$ are linearly independent for all $x \in \Sigma_1 \cap \Sigma_2$. The case that we want to consider in this section is how to define appropriate sliding motion on the intersection $\Sigma_1 \cap \Sigma_2$.

In a neighborhood of the intersection, the phase space consists of four regions S_i, for $i = 1, 2, 3, 4$, with four different vector fields $f_1(x), f_2(x), f_3(x), f_4(x)$. See Figure 1, where we can think to have f_1 when $x \in \{h_1(x) < 0, h_2(x) < 0\}$, f_2 when $x \in \{h_1 < 0, h_2 > 0\}$, f_3 when $\{h_1 > 0, h_2 > 0\}$, and f_4 when $\{h_1 > 0, h_2 < 0\}$. Straightforward extension to this case of Filippov's construction would define the field $f_F(x)$ to be in the convex hull:

$$f_F(x) \in \overline{\text{co}}\,\{f_1(x), f_2(x), f_3(x), f_4(x)\} \ \ : \ f_F(x) = \sum_{i=1}^{4} \alpha_i f_i(x)\,, \tag{4.5}$$

where

$$\sum_{i=1}^{4} \alpha_i = 1, \quad \text{with} \quad 0 \leq \alpha_i \leq 1, \quad \text{for } i = 1, 2, 3, 4\,. \tag{4.6}$$

Sliding motion on $\Sigma_1 \cap \Sigma_2$ would require to have

$$\mathtt{n}_1^T(x) f_F(x) = 0, \quad \mathtt{n}_2^T(x) f_F(x) = 0\,, \tag{4.7}$$

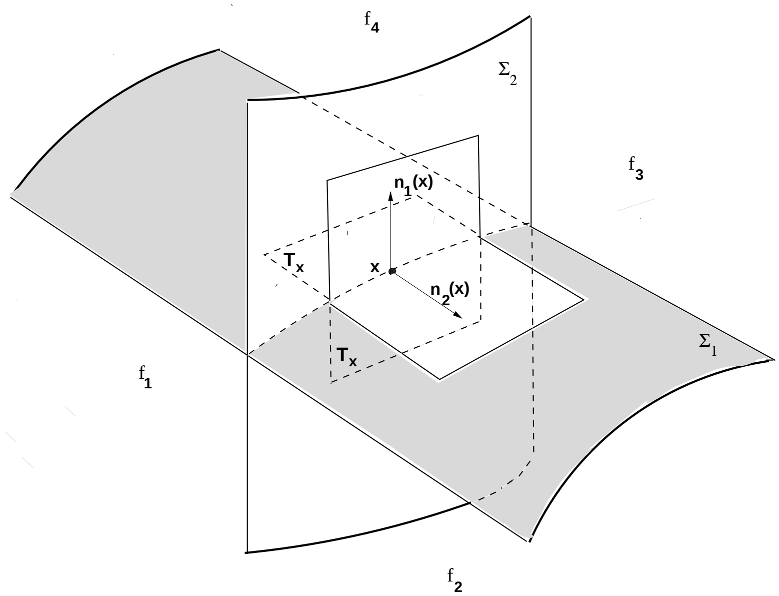

FIGURE 1. Intersecting switching surfaces.

and these last two conditions plus the additional constraint (4.6) will give three equations in four unknowns, with an inherent lack of uniqueness in this construction.

This ambiguity to define a sliding vector field on $\Sigma_1 \cap \Sigma_2$ had already been pointed out in Filippov's book (see [16, p. 52]), and special cases, where the Filippov vector $f_F(x)$ is uniquely defined, were discussed therein, and more recently also in the work [32] where the case of linearly dependent vector fields is considered. Special situations where the ambiguity is bypassed are also in Utkin's books [38, 39] and in Stewart's work [35]. But, in general, the above mentioned ambiguity is not easy to resolve. Several authors have proposed to resolve this ambiguity in different ways. Among the proposed cures, we mention regularization and sigmoid blending techniques. The former is not hard to do in practical cases (such as those arising when the discontinuity naturally arises because of the presence of a sign-function), and it has the advantage of greatly simplifying the theory, but typically leads to very stiff differential equations to solve and a hard problem to tackle numerically (see [27, 28, 29]). The latter technique (sigmoid blending) was introduced in [2, 3], and also re-derived more recently in [12] from the point of view of complementarity systems (see also [19, 30, 33]). In the end, this technique gives a vector field on the intersection, call it f_B, by forming a bilinear interpolant amongst the four vector fields and then imposing the orthogonality conditions:

$$f_B(x) = (1-\beta_1(x))(1-\beta_2(x))f_1(x)+(1-\beta_1(x))\beta_2(x)f_2(x)+\beta_1(x)\beta_2(x)f_3(x)+\beta_1(x)(1-\beta_2(x))f_4(x)\,,$$

where $\beta_{1,2}$ must be found by solving the nonlinear system

$$\mathtt{n}_1^T(x)f_B(x) = 0, \quad \mathtt{n}_2^T(x)f_B(x) = 0\,. \tag{4.8}$$

We want to avoid regularization techniques for several reasons: first, because we do not really understand what regularization means for a general system of the type we are considering, and we are only aware of specific regularization techniques employed on particular problems; second, because –even in the case where these regularization techniques have been employed– one needs to assume that the vector fields $f_i(x)$, $i = 1, 2, 3, 4$, extend outside of the regions where they are naturally defined, and we do not want to assume this property; third, because regularization introduces an error; and, finally, because the non-regularized problems may become simpler numerically, when appropriately discretized (see Section 6). We also want to avoid blending techniques since they are inherently nonlinear (see (4.8)), do not provide a clear geometrical indication of when to leave the intersection itself, and are rather complicated to extend to the case of sliding on the intersection of more than 2 surfaces.

On the other hand, we propose a construction which is a natural extension of Filippov approach for attractive sliding on one surface. Consider $\mathtt{N}(x) = [\mathtt{n}_1(x), \mathtt{n}_2(x)]$ and define the tangent plane $\mathtt{T}_x$ at x on $\Sigma_1 \cap \Sigma_2$ and the projection $\Pi(x)$ onto the tangent plane:

$$\mathtt{T}_x = [\mathtt{n}_1, \mathtt{n}_2]^{\perp}, \qquad \Pi = \mathtt{I} - \mathtt{N}(\mathtt{N}^T\mathtt{N})^{-1}\mathtt{N}^T .$$

(In $\mathbb{R}^3$, we can take $\mathtt{n}(x) = \mathtt{n}_1(x) \times \mathtt{n}_2(x)$, the cross product of the two vectors $\mathtt{n}_1(x)$ and $\mathtt{n}_2(x)$, and then $\mathtt{T}_x = \mathtt{n}(x)\mathtt{n}^T(x)$.)

We define the vector field on $\Sigma_1 \cap \Sigma_2$ by means of a convex combination of the projected vectors $\mathtt{v}_i(x) = \Pi f_i(x)$, for $i = 1, 2, 3, 4$:

$$f_S(x) = \sum_{i=1}^{4} \lambda_i \mathtt{v}_i(x) = \sum_{i=1}^{4} \lambda_i f_i(x) - \sum_{i=1}^{4} \lambda_i \mathtt{N}(\mathtt{N}^T\mathtt{N})^{-1}\mathtt{N}^T f_i(x) \ , \tag{4.9}$$

with $\lambda_i \geq 0$, for $i = 1, 2, 3, 4$, obviously depending on x, and $\sum_{i=1}^{4} \lambda_i = 1$, but the λ_i's are otherwise not yet defined.

Remark 4.1. Regardless of the ambiguity in choosing the values α_i's, Filippov's definition (4.5) would give a vector field $f_F(x)$ on $\Sigma_1 \cap \Sigma_2$ such that

$$\sum_{i=1}^{4} \alpha_i \begin{bmatrix} \mathtt{n}_1^T \\ \mathtt{n}_2^T \end{bmatrix} f_i(x) = \sum_{i=1}^{4} \alpha_i \mathtt{N}^T f_i(x) = 0 \ , \quad x \in \Sigma_1 \cap \Sigma_2 \tag{4.10}$$

from which

$$\sum_{i=1}^{4} \alpha_i \mathtt{N}(\mathtt{N}^T\mathtt{N})^{-1}\mathtt{N}^T f_i(x) = \mathtt{N}(\mathtt{N}^T\mathtt{N})^{-1} \sum_{i=1}^{4} \alpha_i \mathtt{N}^T f_i(x) = 0 \, . \tag{4.11}$$

Thus, if in (4.9) we take $\lambda_i = \alpha_i$, then (4.11) holds and $f_S(x) = f_F(x)$. In other words, our approach (4.9) includes Filippov's choice. At the same time, finding a solution $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, of (4.6) and (4.10), is not an easy problem in general.

Remark 4.2. We ought to point out that not only (see Remark above) Filippov's vector field fits into the model (4.9), and therefore also all techniques which manage to find a vector field in the sense of Filippov (e.g., see [35]), but also other formulations of sliding motion on the intersection do, like the one in [38, 39]. In fact, we believe that (4.9) is sufficiently general to include all cases of possible sliding motion. However, it is not clear how one should find the λ_i's.

We now propose a way to define the λ_i's in (4.9). To begin with, let us consider the implications of having attractive sliding motion. Let us define

$$y_i(x) = \begin{bmatrix} y_{i1}(x) \\ y_{i2}(x) \end{bmatrix} = \begin{bmatrix} \mathtt{n}_1^T f_i(x) \\ \mathtt{n}_2^T f_i(x) \end{bmatrix} = \mathtt{N}^T f_i(x), \qquad i = 1, 2, 3, 4 \ .$$

With respect to the labeling of Figure 1, the conditions to have attractive sliding motion are:

$$y_1 = \begin{pmatrix} y_{11} > 0 \\ y_{12} > 0 \end{pmatrix}, \quad y_2 = \begin{pmatrix} y_{21} > 0 \\ y_{22} < 0 \end{pmatrix}, \quad y_3 = \begin{pmatrix} y_{31} < 0 \\ y_{32} < 0 \end{pmatrix}, \quad y_4 = \begin{pmatrix} y_{41} < 0 \\ y_{42} > 0 \end{pmatrix} \ . \tag{4.12}$$

Next, we proceed in line with our discussion relatively to sliding on one surface, which in particular led us to choose β as in (4.3). First of all, we will want to satisfy *the first order exit conditions*:

$$\mathtt{v}_i = f_i \Rightarrow \lambda_i = 1 \ , \ i = 1, \ldots, 4 \ .$$

Secondly, we would like to take the λ_i's in the form of simple rational functions, which interpolate the exit conditions. We propose the following construction. Define the coefficients μ_i's:

$$\begin{aligned} \mu_1 &= \frac{(a_2^T y_2)(a_3^T y_3)(a_4^T y_4)}{(a_2^T y_2)(a_3^T y_3)(a_4^T y_4) - (a_1^T y_1)} \ , \quad \mu_2 = \frac{(a_1^T y_1)(a_3^T y_3)(a_4^T y_4)}{(a_1^T y_1)(a_3^T y_3)(a_4^T y_4) - (a_2^T y_2)} \ , \\ \mu_3 &= \frac{(a_1^T y_1)(a_2^T y_2)(a_4^T y_4)}{(a_1^T y_1)(a_2^T y_2)(a_4^T y_4) - (a_3^T y_3)} \ , \quad \mu_4 = \frac{(a_1^T y_1)(a_2^T y_2)(a_3^T y_3)}{(a_1^T y_1)(a_2^T y_2)(a_3^T y_3) - (a_4^T y_4)} \ , \end{aligned} \tag{4.13}$$

where the vectors $a_i \in \mathbb{R}^2$, for $i = 1, 2, 3, 4$, will be chosen such that

$$a_1^T y_1 > 0 \ , \quad a_2^T y_2 < 0 \ , \quad a_3^T y_3 < 0 \ , \quad a_4^T y_4 < 0 \ . \tag{4.14}$$

From (4.13) and (4.14), we observe that $0 \le \mu_i \le 1$ for $i = 1, 2, 3, 4$. Moreover, if for some $i \in \{1, 2, 3, 4\}$, we have $\mu_i = 1$, then $\mu_j = 0$ for $j \neq i$, as we desired. However, in general we will have $\sum_{i=1}^4 \mu_i > 1$, instead of being equal to 1, and for this reason we define

$$f_S(x) = \sum_{i=1}^4 \lambda_i \mathtt{v}_i(x) \,, \quad \text{where} \quad \lambda_i = \frac{\mu_i}{\sum_{i=1}^4 \mu_i} \ , \ i = 1, 2, 3, 4 \,, \tag{4.15}$$

which provides our definition for the sliding vector field on $\Sigma_1 \cap \Sigma_2$.

To choose the vectors a_i's, we must enforce $0 \le \mu_i \le 1$, for $i = 1, 2, 3, 4$, and we want that if some $\mu_i = 1$ then $\mu_j = 0$ for $j \neq i$. Subject to these constraints, there is still freedom to choose specific forms for the vectors a_1, a_2, a_3, a_4. The situation is similar to what we already had in choosing a in (4.3). A simple choice, according to (4.14), would be

$$a_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \ , \quad a_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \ , \quad a_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \ , \quad a_4 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \ , \tag{4.16}$$

but with this choice the values of μ_i's in (4.13) may be poorly scaled. A better choice, which gives the same weight to the μ_i's, is to take the a_i's as in (4.16) and then select the μ_i's as in

$$\mu_i = \frac{\left[\prod_{j=1, j\neq i}^4 a_j^T y_j\right]^{1/3}}{\left[\prod_{j=1, j\neq i}^4 a_j^T y_j\right]^{1/3} - a_i^T y_i} \quad i = 1, \ldots, 4 \ . \tag{4.17}$$

This is the choice we have used in our numerical experiments. To sum it up, we choose the vector field on the intersection $\Sigma_1 \cap \Sigma_2$ according to

$$x \in \Sigma_1 \cap \Sigma_2 \ \ : \quad f_S(x) = \sum_{i=1}^4 \lambda_i \mathtt{v}_i(x) \ , \ \ \lambda_i = \frac{\mu_i}{\sum_{i=1}^4 \mu_i} \ , \ \ i = 1, 2, 3, 4 \ ,$$

with μ_i's as in (4.17) and the a_i's as in (4.16).

Remark 4.3. In the case of one surface we obtain $\mu_1 = 1 - \beta$, $\mu_2 = \beta$ in our previous definition of f_S with $\frac{a_1}{a_2} = a$ (see (4.3)).

Example 4.4. Consider the following system in $\mathbb{R}^3$, from [38, p. 64]:

$$\begin{cases} x_1' = u_1 \\ x_2' = u_2 \\ x_3' = u_1 u_2 \end{cases} \ , \tag{4.18}$$

with the two discontinuous controls u_1 and u_2:

$$u_1 = \begin{cases} +1 & \text{when } x_1 < 0 \\ -1 & \text{when } x_1 > 0 \end{cases}, \qquad u_2 = \begin{cases} +1 & \text{when } x_2 < 0 \\ -1 & \text{when } x_2 > 0 \end{cases}.$$

We have the discontinuity surfaces $\Sigma_1 = \{(x_1, x_2, x_3) : x_1 = 0\}$ and $\Sigma_2 = \{(x_1, x_2, x_3) : x_2 = 0\}$. By using Filippov's theory, that is using (4.5)-(4.7), one finds $\alpha_1 = \alpha_4 = \beta$, $\alpha_2 = \alpha_3 = 1/2 - \beta$, and $0 \le \beta \le 1/2$, from which the sliding motion on the intersection $\Sigma_1 \cap \Sigma_2$ is given by the following differential equation

$$\begin{cases} x_1' = 0 \\ x_2' = 0 \\ x_3' = A \end{cases},$$

where A is an undefined value in $[-1, 1]$. [The same "ambiguous" vector field is obtained by using Utkin's equivalent control construction, see [38].] Our approch, instead, yields the value $A = 0$ above, which coincides with the solution α of minimal two-norm in the Filippov's vector field. This means that, with our approach, when we enter $\Sigma_1 \cap \Sigma_2$ at the point $(0, 0, \overline{x}_3)$ we remain there: every entry point $(0, 0, \overline{x}_3)$ is an equilibrium. If we start with initial conditions $x_1(0) = x_2(0)$, then we slide on the line $x_1 = x_2$ until we hit $\Sigma_1 \cap \Sigma_2$ at $(0, 0, \overline{x}_3)$ and then we remain at this point. If we start with $x_1(0) \ne x_2(0)$, we first meet Σ_1 (or Σ_2), then we slide on Σ_1 (or Σ_2) and hit $\Sigma_1 \cap \Sigma_2$ at the intersection point $(0, 0, \overline{x}_3)$.
Notice that if we replace the last differential equation in (4.18) with

$$x_3' = u_1 u_2 + 1 ,$$

then after reaching the intersection $\Sigma_1 \cap \Sigma_2$, we will slide on it with increasing values of x_3.

Remark 4.5. With respect to other approaches, our choice of vector field f_S on the intersection of discontinuity surfaces appears to present some practical advantages. It is very simple, and does not require to solve any non-linear system. It does not need to assume that the fields $f_i(x)$, $i = 1, 2, 3, 4$, extend outside of the regions where they are naturally defined. It leads to a natural 1st order theory which gives clear indication of when (and how) to leave the intersection itself. Morevoer, it extends rather easily to the case of sliding on the intersection of more than 2 surfaces, as we will see below. Nevertheless, a complete theoretical justification of our approach remains to be done.

Let us generalize the approach in (4.15)-(4.16)-(4.17) to the case of several intersecting surfaces. Let us suppose to have p surfaces in $\mathbb{R}^n$, $\Sigma_1, \Sigma_2, \dots, \Sigma_p$, with $n \ge p + 1$. Let each surface be characterized as the zero set of a scalar function $h_i(x)$, $i = 1, \dots, p$, and thus each surface Σ_i is $(n-1)$-dimensional. Assume to have well defined, smooth, gradient vectors $\nabla h_i(x) \ne 0$, for $x \in \bigcap_{i=1}^p \Sigma_i$, that these vectors are linearly independent, and let $\mathtt{n}_1(x), \mathtt{n}_2(x), \dots, \mathtt{n}_p(x)$ be the unit normal vectors. Set $\mathtt{N} = [\mathtt{n}_1, \mathtt{n}_2, \dots, \mathtt{n}_p]$ and write $\Pi = \mathtt{I} - \mathtt{N}(\mathtt{N}^T\mathtt{N})^{-1}\mathtt{N}^T$ for the projection onto $\mathtt{N}^\perp$. Locally, the intersection of these p surfaces divides the space $\mathbb{R}^n$ in 2^p vector fields f_i, $i = 1, \dots, 2^p$. We will define the sliding vector field on the intersection $\bigcap_{i=1}^p \Sigma_i$ as:

$$f_S(x) = \sum_{i=1}^{2^p} \lambda_i \mathtt{v}_i(x) , \quad \mathtt{v}_i = \Pi f_i(x) \ , \ \ i = 1, \dots, 2^p .$$

To determine the λ_i's, just as before, let $y_i = \mathtt{N}^T f_i$, for $i = 1, \dots, 2^p$. Again, the first order conditions to have attractive sliding mean that we have that the components of the vectors y_i, $i = 1, \dots, 2^p$, take all signs patterns in the 2^p combinations of $[\pm 1, \pm 1, \dots, \pm 1]$. E.g., for appropriate ordering of the vector fields, hence of the y_i's, we can assume that all components of y_1 are positive, the second component of y_2 is negative all the others being positive, etc.

Now, for $i = 1, \ldots, 2^p$, let $a_i \in \mathbb{R}^p$ be defined by $a_i = [\pm 1, \pm 1, \ldots, \pm 1]^T$, with the signs chosen so that $\mathrm{sgn}(a_i)_j = -\,\mathrm{sgn}(y_i)_j < 0$ for $i = 2, \ldots, 2^p$, $j = 1, \ldots, p$, and the signs of the entries of a_1 being the same as the signs of the entries of y_1. In particular, this will give $a_i^T y_i < 0$, $i = 2, \ldots, 2^p$, and $a_1^T y_1 > 0$. Then we set

$$\mu_i = \frac{\left[\prod_{j=1, j\neq i}^{2^p} a_j^T y_j\right]}{\left[\prod_{j=1, j\neq i}^{2^p} a_j^T y_j\right] - a_i^T y_i} \ , \ \ i = 1, \ldots, 2^p \ ,$$

and observe that $0 \le \mu_i \le 1$, $i = 1, \ldots, 2^p$, and that if any $\mu_i = 1$, then $\mu_j = 0$, $j \neq i$. But, again, $\sum \mu_i$ could be greater than 1. Thus, we define the sliding vector field on the intersection $\bigcap_{i=1}^p \Sigma_i$ as:

$$f_S(x) = \sum_{i=1}^{2^p} \lambda_i \mathtt{v}_i(x) \,, \quad \lambda_i = \frac{\mu_i}{\sum_{j=1}^{2^p} \mu_j} \ , \ i = 1, \ldots, 2^p \,. \tag{4.19}$$

However, we prefer to work with a more balanced choice for the parameters μ_i's, obtained by choosing

$$\mu_i = \frac{[\prod_{j=1, j\neq i}^{2^p} a_j^T y_j]^m}{[\prod_{j=1, j\neq i}^{2^p} a_j^T y_j]^m - a_i^T y_i} \ , \ \ i = 1, \ldots, 2^p \ , \ \ m = \frac{1}{2^p - 1} \ .$$

Choosing the vectors $a_i = (\pm 1 \ldots, \pm 1)^T$ for $i = 1, \ldots, 2^p$, then

$$\mu_i = \frac{\left[\prod_{j=1, j\neq i}^{2^p} \|y_j\|_1\right]^m}{\left[\prod_{j=1, j\neq i}^{2^p} \|y_j\|_1\right]^m + \|y_i\|_1}, \qquad i = 1, \ldots, 2^p \,, \tag{4.20}$$

where $\|\cdot\|_1$ is the 1-norm on vectors. To summarize, we propose to choose the vector field on the intersection $\Sigma_1 \cap \Sigma_2 \cap \cdots \cap \Sigma_p$ according to (4.19) with with μ_i's as in (4.20).

4.3. **Exiting the intersection onto sliding motion.** Our choice for f_S on $\Sigma_1 \cap \Sigma_2$, reflected exit conditions, embedded in the choice of the values of the μ_i's, which rested on the realization that if one of the vector fields f_i, $i = 1, \ldots, 4$, had lied on the tangent plane, then the trajectory would have exited $\Sigma_1 \cap \Sigma_2$ to enter in S_1, or S_2, or S_3, or S_4. Naturally, it could very well happen that a trajectory will leave $\Sigma_1 \cap \Sigma_2$ while still remaining in Σ_1 or Σ_2. Let us look at what means for the solution to leave the intersection $\Sigma_1 \cap \Sigma_2$ remaining in Σ_1.

On the intersection, we generally have

$$x' = \lambda_1 \mathtt{v}_1 + \lambda_2 \mathtt{v}_2 + \lambda_3 \mathtt{v}_3 + \lambda_4 \mathtt{v}_4 \,, \tag{4.21}$$

that is the sliding vector field feels all four neighboring fields. However, a solution which exits smoothly in Σ_1 will only feel two of the vector fields; in our notation (see Figure 1), either f_1 and f_4 or f_2 and f_3. In other words, defining $\mathtt{z}_i = (I - \mathtt{n}_1 \mathtt{n}_1^T) f_i$, $i = 1, \ldots, 4$, a sliding vector field on Σ_1 will have to be of the form

$$x' = \mu \mathtt{z}_1 + (1-\mu)\mathtt{z}_4, \quad \text{if} \quad y_1 = \begin{pmatrix} y_{11} > 0 \\ y_{12} = 0 \end{pmatrix}, \quad y_4 = \begin{pmatrix} y_{41} < 0 \\ y_{42} = 0 \end{pmatrix}, \tag{4.22}$$

or, respectively,

$$x' = \mu \mathtt{z}_2 + (1-\mu)\mathtt{z}_3, \quad \text{if} \quad y_2 = \begin{pmatrix} y_{21} > 0 \\ y_{22} = 0 \end{pmatrix}, \quad y_3 = \begin{pmatrix} y_{31} < 0 \\ y_{32} = 0 \end{pmatrix} \ . \tag{4.23}$$

In other words, at the point where the trajectory leaves the intersection, we must have had (2nd order conditions)

$$\mathtt{n}_2^T f_1 = \mathtt{n}_2^T f_4 = 0 \left[\text{ hence } \mathtt{z}_1 = \mathtt{v}_1 \ , \ \ \mathtt{z}_4 = \mathtt{v}_4 \ \right] , \tag{4.24}$$

or

$$\mathtt{n}_2^T f_2 = \mathtt{n}_2^T f_3 = 0 \left[\text{ hence } \mathtt{z}_2 = \mathtt{v}_2 \ , \ \ \mathtt{z}_3 = \mathtt{v}_3 \ \right] . \tag{4.25}$$

We summarize by saying that, whenever either (4.24) or (4.25) is satisfied, the solution will slide on Σ_1 with vector field defined by (4.22) or (4.23), respectively. Of course, similar considerations apply to a trajectory which leaves $\Sigma_1 \cap \Sigma_2$ and slides on Σ_2.

5. A numerical method

In this section we consider a numerical approach for PWS-systems. In the introduction, we observed that regularizing or smoothing the system, whenever this is easy to do, does lead to simplifications in the theory. However, small integration steps are usually required during the numerical simulation of the regularized system due to the large derivatives that replace the changes in the structure of the system. Also, it has been observed that regularization may also lead to losing the characteristics of the original phenomenon we are trying to model in the first place (see [27, 28]). Finally, we are not willing to assume that f_1 (say) extend smoothly outside of $S_1 \cup \Sigma$, unlike most works on this subject (see [14, 31, 34]). Therefore, we have adopted a computational approach in which each particular state of the system is integrated with an appropriate numerical method, and the locations where structural changes in the system occur are located in an accurate way; a similar methodology was hinted at in the book [21, p.198]. In [1], this approach is called an *event driven* method. And, just as in other event-driven methods, the numerical method we consider will be effective if there are not too many events, as otherwise the method may become inefficient. In particular, the method makes sense only if on the time interval of interest, say $[0, T]$, there are finitely many event points.

The discussion in this section is done with reference to the model problem (2.1), and we will use the notation therein. We will be mainly concerned with developing a numerical procedure which will accomplish the following different tasks: (i) Integration outside Σ; (ii) Accurate location of points on Σ reached by a trajectory; (iii) Check of the transversality or sliding conditions at the points on Σ; (iv) Integration on Σ (sliding mode); (v) Check of exit conditions and decision of whether or not we should leave Σ. In the exposition below, we will discuss (iii) and (v) relatively to the 1st order conditions only; appropriate generalizations will be pointed out as well.

For discretizing the differential equations, we consider the explicit midpoint rule, which is a 2nd order Runge-Kutta (RK for short) scheme with a simple continuous extension of the same order. The extension itself is useful to find the event points, that is the entry or exit points to the surface. The basic scheme for $\dot{x} = f(x)$, with a general vector field f, with stepsize τ, and given an initial value x_0, has the form

$$x_1 \ = \ x_0 + \tau f(x_{02}) \ , \quad x_{02} = x_0 + \frac{\tau}{2} f(x_0) \ , \tag{5.1}$$

and the second order continuous extension (e.g., see [6, p.125]) is

$$x_1(\sigma) \ = \ x_0 + \sigma[(1 - \frac{\sigma}{\tau}) f(x_0) + \frac{\sigma}{\tau} f(x_{02})] \ , \quad \forall \sigma \in [0, \tau] \ . \tag{5.2}$$

Although it is surely possible to consider other discretization schemes, we stick with explicit one-step schemes, since this allows to evaluate the vector field f_1 (or f_2) only at points where it is well defined. Explicit multistep schemes can also be used, but if the DHR-system has frequent changes in its structure (as in case of chattering systems) one-step methods will be more effective than multistep ones (see [34]).

Remark 5.1. In [38, p.31], an argument is given where it is shown that (for the case of one attracting discontiuity surface), the usual Filippov vector field is obtained in the limit of the stepsize going to 0 for the explicit Euler iterates. [Of course, this argument lends support to use

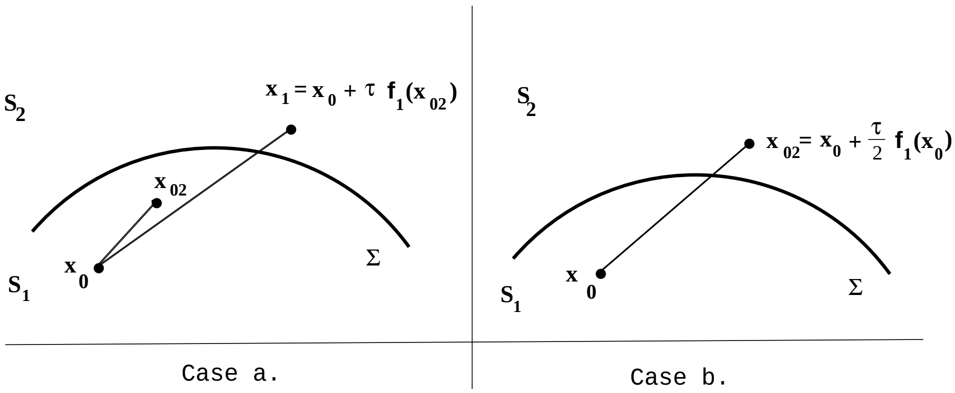

FIGURE 2. Different cases for the midpoint method.

of the Filippov vector field in the case of a single attracting discontinuity surface, not to the use of Euler method!] The case of the intersection of two or more surfaces appears much harder, even with Euler method (and we do not know of anyone having addressed it). Our choice of scheme was purposedly made so to avoid going above/below an attractive discontinuity surface, and as such it does not lend itself to an interpretation similar to the cited one for Euler method. However, if we are willing to use just the explicit midpoint scheme, and to allow for stepping above/below the discontinuity surface, then it is possible to repeat the analysis of Euler method for the explicit midpoint scheme.

5.1. **Integration outside** Σ**.** Integration of (2.1) while the solution remains in S_1, or S_2, is not different than standard numerical integration of a smooth differential system. Therefore, the only interesting case to consider is when, while integrating (say) the system with f_1, we end up reaching/crossing the surface Σ.

So, suppose we are integrating $x'(t) = f_1(x(t))$, starting with $x(0) = x_0$ below the surface Σ, $h(x_0) < 0$, but close to it. Using (5.1), if $h(x_0 + \frac{\tau}{2} f_1(x_0)) < 0$ and also $h(x_1) < 0$, we continue integrating this system. Otherwise, we will have to distinguish between the two cases (see Figure 2):

(a) $h(x_{02}) < 0$ but $h(x_0 + \tau f_1(x_{02})) > 0$, or
(b) $h(x_{02}) > 0$.

Case (a) above is simpler to deal with. Using (5.2), we consider the function $h(x_1(\sigma))$, $\sigma \in [0, \tau]$. This is a continuous function, taking values of opposite sign at the endpoints $\sigma = 0$ and $\sigma = \tau$. Therefore, it has a zero, which we can find by standard techniques, say at $\bar{\sigma}$. This zero will define the point $x_1(\bar{\sigma}) \in \Sigma$. From here, we will need to decide how to proceed, see Sections 5.3 and 5.4.

The case (b) above is instead more elaborate, since the stage value $x_{02}(\tau)$ is already on the other side of Σ, and thus we cannot properly form x_1. In this case, we seek a value $\eta \in (0, \tau)$ such that $x_{02}(\eta) \in \Sigma$ and such that $x_1(\eta) = x_0 + \eta f_1(x_{02}(\eta))$ is on the opposite side of Σ with respect to x_0. This will be possible under some mild (and natural) assumptions.

To begin with, observe that it is always possible to approach the surface Σ "from below". That is, we can assume to have values x_0 where $h(x_0) < 0$, and τ sufficiently small so that if $h(x_0 + \frac{\tau}{2} f_1(x_0)) > 0$, then there is a unique value of $\eta < \tau$ for which $h(x_0 + \frac{\eta}{2} f_1(x_0)) = 0$. With this restriction, we have the following result.

Theorem 5.2. *Let x_0 be given such that $h(x_0) < 0$. Let $\tau > 0$, be sufficiently small, and define $x_{02}(\sigma) := x_0 + \frac{\sigma}{2} f_1(x_0)$, $\sigma \in [0, \tau]$. Suppose that*

(S1) *$h(x_{02}(\tau)) > 0$;*
(S2) *there exists a unique $\eta \in (0,\tau)$ for which $h(x_{02}(\eta)) = 0$;*
(S3) *for the value η of which in* (S2), *$h_x^T(x_{02}(\eta))\ f_1(x_{02}(\eta)) > 0$.*

Then, for η of which in (S2), *we have $h(x_0 + \eta f_1(x_{02}(\eta)) > 0$.*

Proof. We consider the expansion for

$$h(x_0 + \eta f_1(x_{02}(\eta))) = h(x_0 + \eta f_1(x_0 + \frac{\eta}{2} f_1(x_0)))\,.$$

Letting

$$z = \frac{\eta^2}{4}\int_0^1 Df_1(x_0 + s\frac{\eta}{2}f_1(x_0))f_1(x_0)ds + \frac{\eta}{2}f_1(x_0 + \frac{\eta}{2}f_1(x_0))\,,$$

we can rewrite

$$h(x_0 + \eta f_1(x_{02}(\eta))) = h(x_{02}(\eta) + z) = h(x_{02}(\eta)) + \int_0^1 h_x^T(x_{02}(\eta)) + vz)dv \cdot z =$$

$$= \int_0^1 h_x^T(x_{02}(\eta) + vz)dv\big[\frac{\eta}{2}f_1(x_{02}(\eta)) + \frac{\eta^2}{4}\int_0^1 Df_1(x_{02}(s\eta))f_1(x_0)ds\big] = \frac{\eta}{2}[h_x^T(y)f_1(y) + O(\eta)]\ .$$

Hence, because of Assumption (S3), $h(x_0 + \eta f_1(x_0 + \frac{\eta}{2}f_1(x_0))) > 0$ and the result follows. □

In the situation of Theorem 5.2, the previous case (b) can be dealt with. In fact, using (5.2), we consider the function $h(x_1(\sigma))$, $\sigma \in [0,\eta]$. This continuous function changes sign at the endpoints $\sigma = 0$ and $\sigma = \eta$ and thus it has a zero at some value $\bar\sigma$, which will define the point $x_1(\bar\sigma) \in \Sigma$ from which we will restart.

5.2. **Location of points on the surface Σ.** By examining the cases of Section 5.1, we have to find a root $\bar\sigma$ of the scalar function

$$H(\sigma) = h(x_1(\sigma))\ ,\quad x_1(\sigma)\quad \text{from}\quad (5.2)\ , \tag{5.3}$$

where σ will belong to the interval $(0,\tau)$ or the interval $(0,\eta)$. It is desirable to find this root within machine precision, to ensure that the point $x_1(\bar\sigma)$ is on Σ and and avoid numerical oscillations during integration on Σ. Of course, a simple bisection approach can be used, but we eventually resorted (in order to have a faster convergence) to the secant method:

$$\eta_{k+1} = \eta_k - \frac{(\eta_k - \eta_{k-1})}{H(\eta_k) - H(\eta_{k-1})}H(\eta_k), \qquad k \geq 0, \quad \eta_0 = 0,\ \eta_1 = \tau \quad \text{or}\ \eta_1 = \eta\ .$$

We notice that –by using the continuous extension (5.2)– we avoid computing the vector field f_1 except at points where we did for the original scheme. In particular, we never need to evaluate f_1 at points where h may be positive.

5.3. **Integration on Σ.** Once we have a point x_0 on Σ, we need to decide if we will need to cross Σ or slide on Σ. According to the 1st order conditions, we will check if $g_1(x_0)g_2(x_0)$ is (strictly) positive or negative. If $g_1(x_0)g_2(x_0) > 0$, then we integrate the system:

$$x'(t) = f_2(x(t)), \quad x(0) = x_0\ . \tag{5.4}$$

Instead, if $g_1(x_0)g_2(x_0) < 0$, we will have an attractive sliding mode and integrate the system:

$$x'(t) = f_{S,F}(x(t)), \quad x(0) = x_0\ , \tag{5.5}$$

where with $f_{S,F}(x)$ we indicate either the standard Filippov vector field, or the generalization we introduced in (4.1)-(4.3).

Suppose that x_0 is on Σ and that we have a sliding mode solution. When we compute the approximation x_1 of the solution $x(\tau)$ by the explicit midpoint method, in general the vector x_1

will not lie on Σ. To remedy this situation, we project the value x_1 back onto Σ, so to avoid that the numerical solution prematurely leaves the surface $h(x) = 0$. Moreover, even the intermediate stage value $x_0 + \frac{\tau}{2} f_{S,F}(x_0)$ in general will not be on Σ, and thus before computing x_1 we project the stage value onto Σ as well. Succintly, one step of the projected midpoint scheme on Σ is expressed as:

1. $\widehat{x}_{02} = x_0 + \frac{\tau}{2} f_{S,F}(x_0)$;
2. $x_{02} = P(\widehat{x}_{02})$;
3. $\widehat{x}_1(\tau) = x_0 + \tau f_{S,F}(x_{02})$;
4. $x_1(\tau) = P(\widehat{x}_1(\tau))$;

where $P(y)$ denotes the Euclidean projection onto the tangent space at Σ_y. In a similar way, we define the projected continuous extension of the method as

$$x_1(\sigma) \;=\; P\big(x_0 + \sigma[(1-\frac{\sigma}{\tau})f_1(x_0) + \frac{\sigma}{\tau} f_1(x_{02})]\big) \ , \quad \sigma \in [0,\tau] \ , \tag{5.6}$$

where it is understood that the value x_{02} is the projected value.

It is worth observing that the projection operator does not change the overall order of the method which remains 2. [Of course, if $h(x)$ is linear with respect to x, that is if Σ is flat, then no projection is required because the numerical solution $x_1(\tau)$ will automatically remain on Σ.] The issue of how to do the projection, and its associated expense, is discussed in Section 5.5.

While we integrate on Σ, we will monitor if we have to continue sliding on it, or if we need to leave Σ.

5.4. **Exit conditions.** Once the point x_1 on Σ has been computed, we need to check the first order exit conditions. That is, if $g_1(x_1)g_1(x_0) < 0$ or $g_2(x_1)g_2(x_0) < 0$. If neither of these is true, we continue integrating on Σ.

To fix ideas, suppose, instead, that $g_1(x_1)g_1(x_0) < 0$. In this case, we seek a zero of the function

$$g_1(x_1(\sigma)) \ , \quad \sigma \in [0,\tau] \ , \quad \text{with} \quad x_1(\sigma) \quad \text{from (5.6)} \ ;$$

notice that the function $g_1(x_1(\sigma))$ depends continuously on σ and changes sign at the endpoints. As before, we have used a secant method to find a root. Once this zero is found, sat at $\bar{\sigma}$, we will leave Σ and proceed integrating in S_1 (assuming that $g_2(x_1(\bar{\sigma})) > 0$). Similar reasoning applies of course if it is g_2 to change sign at x_0 and x_1.

When the first order exit conditions are not satisfied (say, we find a point where both g_1 and g_2 vanish), we need to look at higher order conditions, see Section 3.

5.5. **Projection on Σ.** The projection on Σ is done in the standard way (e.g., see [14, 20]), with some simplifications due to the specific nature of our problem.

If $\widehat{x}$ is a point close to Σ, then the projected vector $x = P(\widehat{x})$ on Σ is the solution of the following constrained minimization problem

$$\min_{x\in\Sigma} g(x) \ , \quad g(x) = \frac{1}{2}(\widehat{x}-x)^T(\widehat{x}-x) \ .$$

By using the Lagrange's multiplier's method, we have to find the root of

$$G(x,\lambda) = \begin{pmatrix} \nabla g(x) + \lambda \nabla h(x) \\ h(x) \end{pmatrix},$$

where $\lambda \in \mathbb{R}$. Consider Newton's method to compute the root of $G(x,\lambda)$:

$$G'(x^k,\lambda^k)\begin{pmatrix} \Delta x^k \\ \Delta\lambda^k \end{pmatrix} = -G(x^k,\lambda^k), \quad k \geq 0,$$

where $\Delta x^k = x^{k+1} - x^k$, $\Delta\lambda^k = \lambda^{k+1} - \lambda^k$, for $k \geq 0$, and

$$G'(x,\lambda) = \begin{pmatrix} I + \lambda h_{xx}(x) & \nabla h(x) \\ \nabla^T h(x) & 0 \end{pmatrix}.$$

To avoid having to solve a true linear system at each k, we actually use the following simplified Newton iteration

$$\begin{bmatrix} I & \nabla h(x^k) \\ \nabla^T h(x^k) & 0 \end{bmatrix} \begin{bmatrix} \Delta x^k \\ \Delta \lambda^k \end{bmatrix} = - \begin{bmatrix} \widehat{x} - x^k + \lambda^k \nabla h(x^k) \\ h(x^k) \end{bmatrix} ;$$

this is legitimate, since we expect that the value of λ will be close to 0 and a few iterates are typically needed to converge to the point on Σ. Observe that the linear system we solve has a coefficient matrix with a simple structure and a simple factorization: $\begin{pmatrix} I & b \\ b^T & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ b^T & 1 \end{pmatrix} \begin{pmatrix} I & b \\ 0 & -b^T b \end{pmatrix}$.

Projecting onto the intersection of several surfaces. In case we have sliding motion on the intersection of two (or more) surfaces, the integration will proceed with the usual numerical scheme, and we will need to project points onto the intersection. We accomplish this similarly to the previous case of one surface. To clarify, suppose that we have to project a point $\widehat{x}$ onto $\Sigma_1 \cap \ldots \cap \Sigma_p$, defined by the system $h_1(x) = 0, \ldots, h_p(x) = 0$. Then, we seek the solution of the constrained minimization problem

$$\min_{x \in \Sigma_1 \cap \ldots \cap \Sigma_p} g(x) \ , \quad g(x) = \frac{1}{2} \|\widehat{x} - x\|_2^2 \, ,$$

for which the method of Lagrange's multipliers requires us to find the root of

$$G(x, \lambda_1, \ldots, \lambda_p) = \begin{pmatrix} \nabla g(x) + \lambda_1 \nabla h(x) + \cdots + \lambda_p \nabla h_p(x) \\ h_1(x) \\ \vdots \\ h_p(x) \end{pmatrix},$$

with $\lambda_1, \lambda_2, \ldots, \lambda_p \in \mathbb{R}$. The simplified Newton iteration we use is now given as

$$\begin{bmatrix} I & \nabla h_1(x^k) & \ldots & \nabla h_p(x^k) \\ \nabla^T h_1(x^k) & 0 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ \nabla^T h_p(x^k) & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} \Delta x^k \\ \Delta \lambda^1 \\ \vdots \Delta \lambda_p \end{bmatrix} = - \begin{bmatrix} \widehat{x} - x^k + \lambda_1^k \nabla h_1(x^k) + \cdots + \lambda_p^k \nabla h_p(x^k) \\ h_1(x^k) \\ \vdots \\ h_p(x^k) \end{bmatrix}.$$

Again, the structure of the system is simple, and admits a simple block factorization. In the case of two surfaces, this looks like $\begin{pmatrix} I & b_1 & b_2 \\ b_1^T & 0 & 0 \\ b_2^T & 0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ b_1^T & 1 & 0 \\ b_2^T & 0 & 1 \end{pmatrix} \begin{pmatrix} I & b_1 & b_2 \\ 0 & -b_1^T b_1 & -b_1^T b_2 \\ 0 & -b_2^T b_1 & -b_2^T b_2 \end{pmatrix}$.

6. Numerical Experiments

We show results of numerical simulation on three Examples: (i) A planar problem with a periodic orbit having a sliding mode, (ii) An example where sliding occurs on the intersection of two surfaces, (iii) An example where chattering (or a sliding mode) exist (actually, this is Example 3.2). All our results have been otbained with an experimental program we wrote in `Matlab`, implementing the methods described in the previous Section; a public domain `Matlab` code to solve a class of Filippov's systems is given in [32].

Example 6.1. *Sliding on a line-segment.* This simple example is one which we can understand by hand calculation and it is helpful to illustrate the methods. It is an example in the same flavor of a problem in [27, 28] (the so-called *stick-slip* system). We have the two-dimensional system

$$x' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{cases} f_1(x) , & h(x) < 0 , \\ f_2(x) , & h(x) > 0 , \end{cases}$$

with

$$f_1(x) = \begin{pmatrix} x_2 \\ -x_1 + \frac{1}{1.2 - x_2} \end{pmatrix}, \quad f_2(x) = \begin{pmatrix} x_2 \\ -x_1 - \frac{1}{0.8 + x_2} \end{pmatrix} ,$$

and the surface Σ is defined by the zero set of $h(x) := x_2 - 0.2$. We notice that $\nabla h(x) = [0\ 1]^T$, and thus on Σ we have

$$g_1(x) = \nabla^T h(x) f_1(x) = -x_1 + 1 , \qquad g_2(x) = \nabla^T h(x) f_2(x) = -x_1 - 1 ,$$

and so there will be an attractive sliding mode on Σ when $x \in (-1, 1)$. The projections of the vector fields onto Σ are $v_1 = v_2 = \left[\begin{smallmatrix} x_2 \\ 0 \end{smallmatrix} \right]$ and therefore there is only one possible sliding vector field on Σ, the vector

$$f_S(x) = \begin{pmatrix} x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.2 \\ 0 \end{pmatrix} .$$

Thus, on Σ, the x_1-component of the solution will grow linearly until reaching the value $x_1 = 1$, at which point the trajectory will leave Σ, with vector field f_1. In Figure 3 we show the limit cycle for this problem, a red mark indicating the exit point from Σ, as well as a trajectory (starting at the point marked with a red cross) which reaches the limit cycle through previous crossing of Σ and sliding on it. Obviously, at the points in which we enter the surface Σ there is lack of differentiability of the solution, whereas at the value $x_1 = 1$, the solution leaves the surface differentiably. We also notice that the equilibrium point $(1/1.2, 0)$ for f_1 is inside the limit cycle.

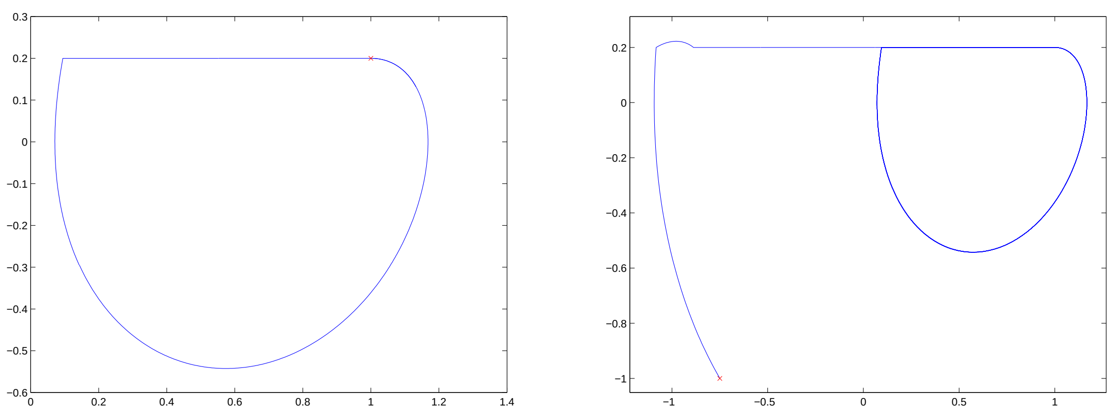

FIGURE 3. Limit cycle with sliding segment (left) for Example 6.1, and approaching it through crossing and sliding (right).

Example 6.2. *Sliding on intersecting surfaces.* Let us consider the three-dimensional system:

$$x'(t) = f(x(t)) \tag{6.1}$$

where $x = (x_1, x_2, x_3)$ and $f(x(t))$ is a discontinuous vector field with respect to the two surfaces:

$$\Sigma_1 \ : \ \{ x : \ h_1(x) = 0 \ , \ \ h_1(x) := x_1 + x_2 + x_3 - Z \ \} \ ,$$

$$\Sigma_2 \ : \ \{x: \ h_2(x)=0 \ , \ h_2(x):=2x_1^2+2(x_2-x_3)^2-R \ \} \ ,$$

with R and Z positive constants. Here, $f(x) := A(S_1,S_2)x + b(S_2)$ where:

$$A(S_1,S_2)=\frac{1}{2}\begin{pmatrix} 2(\gamma_1-k_1S_1) & 2\alpha & -2\alpha \\ (-\alpha-\gamma_1+k_1S_1-\gamma_2) & (-\alpha+\gamma_1-k_1S_1-\gamma_2) & (\alpha-\gamma_1+k_1S_1-\gamma_2) \\ (\alpha-\gamma_1+k_1S_1-\gamma_2) & (-\alpha-\gamma_1+k_1S_1-\gamma_2) & (\alpha+\gamma_1-k_1S_1-\gamma_2) \end{pmatrix},$$

$$b(S_2)=\frac{1}{2}\begin{pmatrix} 0 \\ k_2S_2 \\ k_2S_2 \end{pmatrix} \ ,$$

with $\gamma_1, \gamma_2, \alpha, k_1, k_2$ are positive constants and where

$$S_1=\begin{cases} 0, & h_2(x_1,x_2,x_3)<0 \\ 1, & h_2(x_1,x_2,x_3)>0 \end{cases} \ , \quad S_2=\begin{cases} 0, & h_1(x_1,x_2,x_3)>0 \\ 1, & h_1(x_1,x_2,x_3)<0 \end{cases} \ .$$

Thus, we have four different vector fields in the four regions of $\mathbb{R}^3$ isolated by the two surfaces $h_1(x)$ and $h_2(x)$. Correspondingly, for $f(x)$ we will use the notation:

$f_1(x)$ when $h_1(x)>0$ and $h_2(x)>0$ (region I);
$f_2(x)$ when $h_1(x)>0$ and $h_2(x)<0$ (region II);
$f_3(x)$ when $h_1(x)<0$ and $h_2(x)<0$ (region III);
$f_4(x)$ when $h_1(x)<0$ and $h_2(x)>0$ (region IV).

Observe that

$$\nabla h_1(x)=[1,1,1]^T \ , \qquad \nabla h_2(x)=4\,[x_1, x_2-x_3, x_3-x_2]^T \ ,$$

and thus on Σ_1, respectively Σ_2, we get

$$\mathtt{n}_1=\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} \ , \quad \mathtt{n}_2=\frac{1}{\sqrt{R-x_1^2}}\begin{bmatrix}x_1\\x_2-x_3\\x_3-x_2\end{bmatrix} \ , \quad \mathtt{N}=\begin{bmatrix}\mathtt{n}_1 & \mathtt{n}_2\end{bmatrix} \ .$$

Recalling that $y_i=\mathtt{N}^T f_i$, and letting $D=\mathrm{diag}\big(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{R-x_1^2}}\big)$, we have

$$y_1=D\begin{bmatrix}-\gamma_2 Z\\2(\gamma_1-k_1)R\end{bmatrix}, \quad y_2=D\begin{bmatrix}-\gamma_2 Z\\2\gamma_1 R\end{bmatrix}, \quad y_3=D\begin{bmatrix}-\gamma_2 Z+k_2\\2\gamma_1 R\end{bmatrix}, \quad y_4=D\begin{bmatrix}-\gamma_2 Z+k_2\\2(\gamma_1-k_1)R\end{bmatrix} \ .$$

In order to have sliding on the intersection, we need to satisfy

$$\begin{aligned} \mathtt{n}_1^T(x)f_1(x)<0 &\Rightarrow \tfrac{-\gamma_2 Z}{\sqrt{3}}<0 \ , & \mathtt{n}_1^T(x)f_2(x)<0 &\Rightarrow \tfrac{-\gamma_2 Z}{\sqrt{3}}<0 \ , \\ \mathtt{n}_1^T(x)f_3(x)>0 &\Rightarrow \tfrac{-\gamma_2 Z+k_2}{\sqrt{3}}>0 \ , & \mathtt{n}_1^T(x)f_4(x)>0 &\Rightarrow \tfrac{-\gamma_2 Z+k_2}{\sqrt{3}}>0 \ , \\ \mathtt{n}_2^T(x)f_1(x)<0 &\Rightarrow \tfrac{2(\gamma_1-k_1)R}{\sqrt{R-x_1^2}}<0 \ , & \mathtt{n}_2^T(x)f_2(x)>0 &\Rightarrow \tfrac{2(\gamma_1-k_1)R}{\sqrt{R-x_1^2}}>0 \ , \\ \mathtt{n}_2^T(x)f_3(x)>0 &\Rightarrow \tfrac{2(\gamma_1-k_1)R}{\sqrt{R-x_1^2}}>0 \ , & \mathtt{n}_2^T(x)f_4(x)<0 &\Rightarrow \tfrac{2(\gamma_1-k_1)R}{\sqrt{R-x_1^2}}<0 \ , \end{aligned}$$

and these relations constrain the values of the parameter. In our experiments, we fixed

$$\gamma_1=\gamma_2=Z=1, \qquad k_1=k_2=2,$$

so that the conditions for having attractive sliding motion on the intersection are satisfied.

Finally, we take the vectors a_i as in (4.16):

$$a_1=\begin{bmatrix}1\\1\end{bmatrix}, \quad a_2=\begin{bmatrix}-1\\1\end{bmatrix}, \quad a_3=\begin{bmatrix}1\\1\end{bmatrix}, \quad a_4=\begin{bmatrix}1\\-1\end{bmatrix} \ ,$$

so that:

$$a_1^Ty_1=-\frac{1}{\sqrt{3}}-\frac{2R}{\sqrt{R-x_1^2}}, \quad a_2^Ty_2=\frac{1}{\sqrt{3}}+\frac{2R}{\sqrt{R-x_1^2}}, \quad a_3^Ty_3=\frac{1}{\sqrt{3}}+\frac{2R}{\sqrt{R-x_1^2}}, \quad a_4^Ty_4=\frac{1}{\sqrt{3}}+\frac{2R}{\sqrt{R-x_1^2}} \ .$$

Using (4.17) to compute the weighted coefficients μ_i's, we obtain:

$$\mu_1 = \mu_2 = \mu_3 = \mu_4 = \frac{1}{2} \ , \quad \text{and hence} \quad \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \frac{1}{4} \ ,$$

which will be used in (4.15) to provide the vector field on the intersection.

Remark 6.3. Les us contrast what we just did with the case of Filippov, that is with sliding vector field $f_F(x) = \sum_{i=1}^{4} \alpha_i f_i(x)$. Now we would need to satisfy (4.6)-(4.7), which gives the 1-parameter familiy of solutions

$$\alpha_1 = \alpha_3 \ , \ \ \alpha_2 = \alpha_4 \ , \quad \text{and} \quad \alpha_1 = \frac{1 - 2\alpha_2}{2} \ , \ \ \alpha_2 \in [0, 1/2] \, .$$

The solution $\alpha_i = 1/4$, $i = 1, \dots, 4$, corresponds to the choice for which the vector $\alpha = (\alpha_i)_{i=1}^4$ has minimal 2-norm.

In Figure 4, we show two typical trajectories for this problem. On the left is a situation in which we start from "region I" until we hit the surface Σ_2; we then slide on this, until we hit the intersection of the two surfaces, on which we slide. On the right is a case in which we also start from region I, but we first hit the surface Σ_1, and we slide on it, until we hit the intersection with Σ_2, and slide on it. From the figures, the entry points on the surfaces are clearly distinguishable, and we observe the lack of differentiability of the solution there.

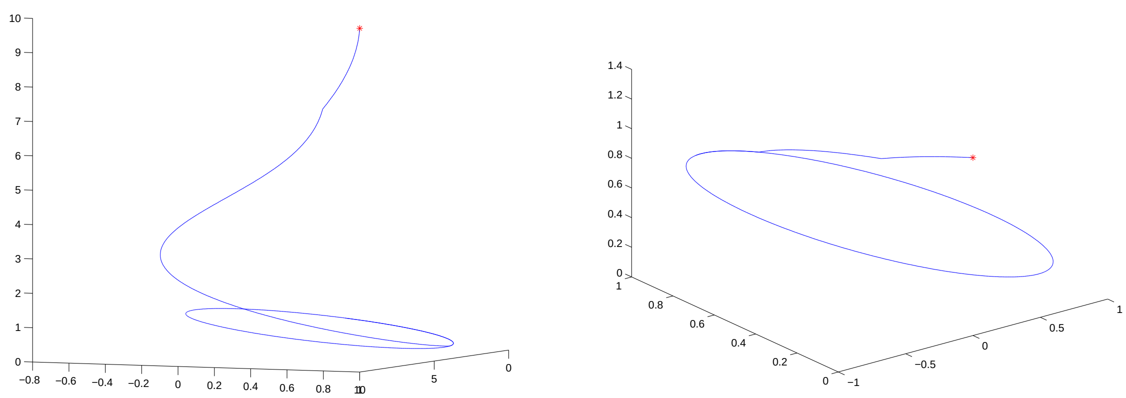

FIGURE 4. Sliding on the intersection of the two surfaces.

Example 6.4 (Chattering)**.** This is Example 3.2:

$$x'(t) = Ax(t) + bu(t) \qquad \text{with} \qquad u = -\text{sign}(y) = \begin{cases} 1 & y < 0 \ , \\ [-1, 1] & y = 0 \ , \\ -1 & y > 0 \ . \end{cases} \tag{6.2}$$
$$y(t) = c^T x(t)$$

We take two concrete instances of this model, by fixing (see [23, 24])

$$A = \begin{bmatrix} -4 & 1 & 0 & 0 \\ -6 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \ , \ c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \ , \quad \text{and (i)} \quad b = \begin{bmatrix} 1 \\ -3\beta \\ 3\beta^2 \\ -\beta^3 \end{bmatrix} \ , \quad \text{or (ii)} \quad b = \begin{bmatrix} 0 \\ 1 \\ -2\beta \\ \beta^2 \end{bmatrix} \ , \ \beta = 1/5 \ .$$

In case (i), we will have a trajectory with a portion on the sliding set of order 1, the plane $x_1 = 0$, in the case (ii) the trajectory will exhibit chattering (frequent switching) about the sliding set of order 2, $x_1 = x_2 = 0$. To elucidate, in Figure 5, on the left we show –in the (x_1, x_2, x_3) space– the periodic trajectory in case (i), and we can appreciate that the solution slides on the plane $x_1 = 0$, twice. On the right of Figure 5 we show the solution components in this case. Obviously, past the transient, the behavior is periodic, and the lack of differentiability when the solution hits the sliding plane is apparent.

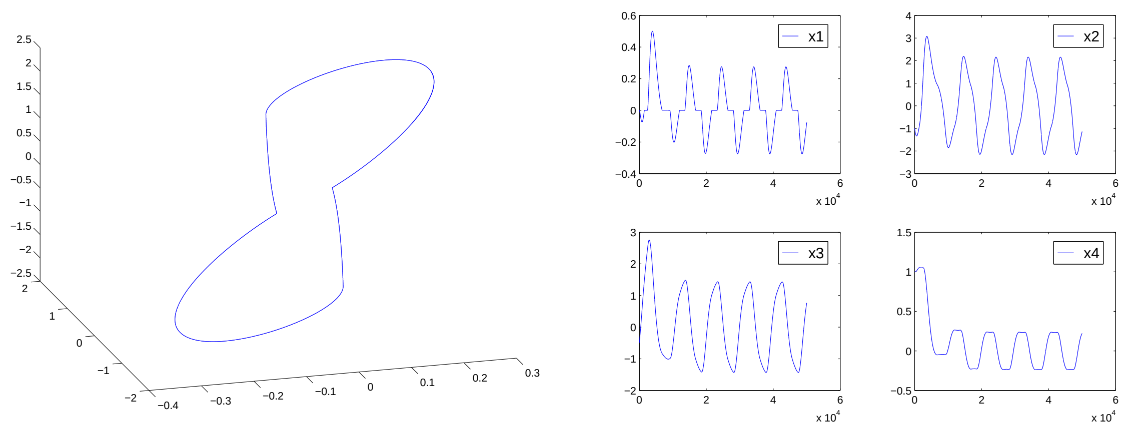

FIGURE 5. Periodic trajectory, and solution components, sliding on the 1st order sliding set $x_1 = 0$.

In Figure 6, we show –in (x_1, x_2, x_3) space– the trajectory in the case of chattering, and the frist two solution components with their enlargement. Observe that the solution appears to slide on the sliding set of order 2, $x_1 = x_2 = 0$, but in fact it does not and it undergoes frequent switching near this set, as it is elucidated in the enlargement.

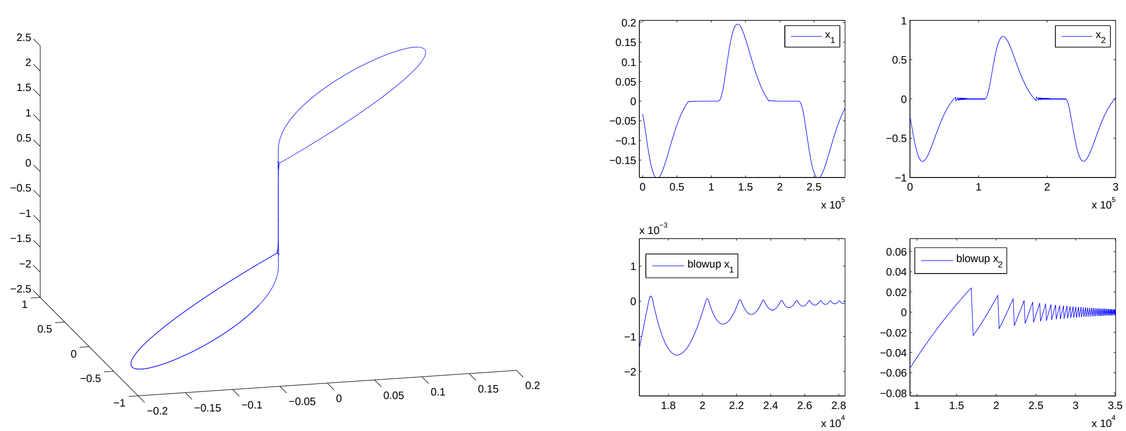

FIGURE 6. Periodic trajectory, chattering about the sliding set of order 2: $x_1 = x_2 = 0$.

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