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Stabilization and tracking in the nonholonomic integrator via sliding modes[1]

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Abstract

In this paper we use an approach based on sliding mode control to design a feedback which stabilizes the origin for the so-called nonholonomic integrator or Heisenberg system, a particular case of a canonical class of nonlinear driftless control systems of the form

$$\dot{x} = B(x)u$$

which fail Brockett's necessary condition for the existence of a smooth stabilizing feedback.

Keywords: Nonholonomic system; Sliding modes; Nonlinear control

1. Introduction

There has been a great deal of research recently on the problem of stabilizing systems which fail a necessary condition for the existence of smooth, or even continuous feedback (see [6]). One of the reasons for the interest in such systems is that nonholonomic systems fall into this class. (See, for example, [21, 4] and references therein.) Various approaches have been taken to the stabilization problem for such systems, focusing mainly on the development of either smooth dynamic feedback or nonsmooth feedback. An important paper regarding the former approach is Coron [9]. See also Pomet [22] and M'Closkey and Murray [20]. Kolmanovsky and McClamroch [17], for example, used a discontinuous, discrete-time approach, while Brockett [7] used a stochastic approach. Other important work includes that of Liu and Sussmann [18], as well as the work of Samson, Sordalen, Walsh, Bushnell and others. Another interesting problem for such systems in the problem of tracking which was also analyzed in Brockett [7].

In this paper we consider a sliding mode approach to the stabilization and tracking problem for the so-called nonholonomic integrator or Heisenberg system (so called because the underlying Lie algebra of control vector fields is isomorphic to the Heisenberg algebra). Firstly we provide a feedback which will globally asymptotically stabilize the origin. The idea is to use the natural algebraic structure of the system together with ideas from sliding mode theory (see [24, 10, 11]). We announced the main result on stabilization in Bloch and Drakunov [1]. Related recent work includes the following. Khennouf and Canudas de Wit [16] presented the alternative control scheme (24), (25), discussed below, and in Canudas de Wit and Khennouf [8], they considered robustness issues.

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Hespanha [15] and Morse [19] discussed these control ideas in terms of logic-based switching. Sliding mode control for differentially flat nonholonomic systems is discussed in Sira-Ramirez [23]. Tracking via sliding modes is described in Guldner and Utkin [14] for holonomic systems.

Further, using a variant of our techniques, we derive an approximate tracker for our system (see also [2]).

The nonholonomic integrator is merely the simplest example of an important class of nonlinear controllable systems of the $\dot{x} = B(x)u$, $B(x)$ an $n \times m$ matrix, $m < n$, introduced in the fundamental paper of Brockett [5], and which are prototypical examples of systems where smooth feedback fails and which have a natural controllability condition. In a related paper [3] we discuss the stabilization of such general systems via a discontinous (but not sliding mode) algorithm. Our algorithm for the general case involves switching between different Lyapunov functions.

2. Stabilization of the nonholonomic integrator in sliding mode

We consider the system (see [5])

$$\dot{x} = u, \tag{1}$$

$$\dot{y} = v, \tag{2}$$

$$\dot{z} = xv - yu. \tag{3}$$

As mentioned earlier it is a prototype for more complex controllable but not smoothly stabilizable systems.

We note also that this system can be obtained by a change of variables from a system in so-called "chained form" (see [23] and references therein).

The problem of stabilizing (1), (2), even locally, is not a trivial task, since, as can be easily seen, the linearization in the vicinity of the origin gives the non-controllable system

$$\dot{x} = u,$$

$$\dot{y} = v,$$

$$\dot{z} = 0.$$

In fact, as was proved by Brockett [6], the system (1)–(3) cannot be stabilized by *any* smooth feedback control law. As discussed in the introduction, later approaches using time-periodic feedback and a randomized feedback were developed.

In this paper we present time-invariant laws solving the stated problem. By nature, they are discontinuous and lead to sliding along manifolds of reduced dimensionality in the state space.

The main difficulty here is the fact that stabilization of x and y leads to zero right-hand side of (3) and, therefore, the variable z cannot be steered to zero. That simple observation implies that to stabilize the system one needs to make z converge "faster" than x and y.

We suggest using the following control law:

$$u = -\alpha x + \beta y \operatorname{sign}(z), \tag{4}$$

$$v = -\alpha y - \beta x \operatorname{sign}(z), \tag{5}$$

where α and β are positive constants.

Let us show that there exists a set of initial conditions such that trajectories starting there converge to the origin.

Consider a Lyapunov function for (x, y)-subspace:

$$V = \tfrac{1}{2}(x^2 + y^2). \tag{6}$$

The time derivative of V along the trajectories of system (1)–(3) is negative:

$$\begin{aligned}\dot{V} &= -\alpha x^2 + \beta xy \operatorname{sign}(z) - \alpha y^2 - \beta xy \operatorname{sign}(z)\\ &= -\alpha(x^2 + y^2) = -2\alpha V.\end{aligned} \tag{7}$$

Therefore, under the control (4), (5) the variables x and y are stabilized.

Now let us consider the variable z. Using (3) and (4), (5) we obtain

$$\begin{aligned}\dot{z} &= xv - yu = -\beta(x^2 + y^2) \operatorname{sign}(z)\\ &= -2\beta V \operatorname{sign}(z).\end{aligned} \tag{8}$$

Since V does not depend on z and is a positive function of time, the absolute value of the variable z will decrease and will reach zero in finite time if the inequality

$$2\beta \int_0^\infty V(\tau)\,\mathrm{d}\tau > |z(0)| \tag{9}$$

holds. If $z(0)$ is such that

$$2\beta \int_0^\infty V(\tau)\,\mathrm{d}\tau = |z(0)|, \tag{10}$$

$z(t)$ converges to the origin in infinite time (asymptotically). Otherwise, it converges to some constant nonzero value of the same sign as $z(0)$.

If the above inequality (9) holds, the system trajectories are directed to the surface $z = 0$ and the variable $z(t)$ it stabilized at the origin in finite time. (The variables x and y, as follows from (7), always converge to the origin while within that surface.)

This phenomenon is known as *sliding mode* (see [24, 10]). The manifold $z = 0$ is a stable integral manifold of the closed loop system (1)–(3), (4), (5). Its characteristic feature is reachability in finite time [11]. Using a smooth control (even a control satisfying a local Lipschitz condition (in the vicinity of $\{z = 0\}$)) such fast convergence cannot be achieved. On the other hand, within the sliding manifold $\{z = 0\}$ the system behavior is described in accordance with the Filippov definition for systems of differential equations with discontinuous right-hand sides [12].

Let us explain the version of this definition which we are using. We consider the system

$$\dot{x} = f(x), \tag{11}$$

with $f(x)$ a discontinuous function comprised of a finite number of continuous $f_k(x)$ $(k = 1, \ldots, N)$ so that

$$f(x) \equiv f_k(x) \quad \text{for } x \in \mathcal{M}_k, \tag{12}$$

where the open regions $\mathcal{M}_k$ have piecewise smooth boundaries $\partial\mathcal{M}_k$. Then according to the Filippov definition we define the right-hand side of (11) within $\partial\mathcal{M}_k$ as

$$\dot{x} = \sum_{k \in I(x)} \mu_k f_k(x). \tag{13}$$

The sum is taken over the set $I(x)$ of all k such that $x \in \partial\mathcal{M}_k$ and the variables μ_k satisfy

$$\sum_{k \in I(x)} \mu_k = 1, \tag{14}$$

i.e. the right-hand side belongs to the convex closure $\mathrm{co}\{f_k(x)\colon k \in I(x)\}$ of the vector fields $f_k(x)$ for all $k \in I(x)$. Actually, the Filippov definition replaces the differential equation (11) by a differential inclusion

$$\dot{x} \in \mathrm{co}\{f_k(x)\colon k \in I(x)\} \tag{15}$$

for the points x belonging to the boundaries $\partial\mathcal{M}_k$. If within the convex closure there exists a vector field tangent to all or some of the boundaries then there is a solution of the differential inclusion belonging to $\partial\mathcal{M}_k$ which corresponds to the sliding mode.

In the above relatively simple case, the Filippov definition provides a unique solution and implies that the system on the manifold is

$$\dot{x} = -x,$$

$$\dot{y} = -y.$$

Since from (7) it follows that

$$V(t) = V(0)\,\mathrm{e}^{-2\alpha t} = \tfrac{1}{2}(x^2(0) + y^2(0))\,\mathrm{e}^{-2\alpha t}, \tag{16}$$

substituting this expression in (9) and integrating we find that the condition for the system to be stabilized is

$$\frac{\beta}{2\alpha}[x^2(0) + y^2(0)] \geqslant |z(0)|. \tag{17}$$

The inequality

$$\frac{\beta}{2\alpha}(x^2 + y^2) < |z|, \tag{18}$$

defines a parabolic region $\mathscr{P}$ in the state space.

The above derivation can be summarized in the following theorem:

Theorem 1. *If the initial conditions for the system* (1)–(3) *belong to the complement* $\mathscr{P}^{\mathrm{c}}$ *of the region* $\mathscr{P}$ *defined by* (18), *then the control* (4), (5) *stabilizes the state.*

If the initial data are such that (18) is true, i.e. the state is inside the paraboloid, we can use any control law which steers it outside. In fact, any nonzero constant control can be applied. Namely, if $u \equiv u_0 =$ const, $v \equiv v_0 =$ const, then

$$x(t) = u_0 t + x_0,$$

$$y(t) = v_0 t + y_0,$$

$$z(t) = t(x_0 v_0 - y_0 u_0) + z_0.$$

With such x, y and z the left-hand side of (18) is quadratic with respect to time t while the right-hand side is linear. Hence, when the time increases the state inevitably will leave $\mathscr{P}$.

A global control law in the form of the feedback (although discontinuous) can be described as follows:

$$(u, v)^{\mathrm{T}} = \begin{cases} (u_0, v_0)^{\mathrm{T}} & \text{if } (x, y, z)^{\mathrm{T}} \in \mathscr{P}, \\ \text{Eqs. (4), (5)} & \text{if } (x, y, z)^{\mathrm{T}} \in \mathscr{P}^{\mathrm{c}}. \end{cases} \tag{19}$$

Theorem 2. *The closed system* (1)–(3),(19) *is globally asymptotically stable at the origin.*

Global asymptotic stability means that: (i) for all initial conditions, $x(t), y(t), z(t) \to 0$, when $t \to \infty$;

(ii) $\forall\varepsilon > 0$ there exists $\delta > 0$ such that $x_0^2 + y_0^2 + z_0^2 < \delta^2$ implies $x^2(t) + y^2(t) + z^2(t) < \varepsilon^2$ for any $t \geqslant 0$ (Lyapunov stability).

We have already shown above that (i) is true. (ii) follows from the fact that outside $\mathscr{P}$ and on the surface of parabola $\partial\mathscr{P}$ the state monotonically approaches the origin. For initial conditions inside $\mathscr{P}$ we have

$$x^2(t) + y^2(t) + z^2(t) = (u_0 t + x_0)^2 + (v_0 t + y_0)^2 + [(x_0 v_0 - y_0 u_0)t + z_0]^2 \tag{20}$$

The maximum of the expression (20) is achieved for $t = 0$ or $t = t_f$, where t_f is the first moment of time when the state reaches $\partial\mathscr{P}$. This moment is defined by an equation

$$\frac{\beta}{2\alpha}(u_0 t_f + x_0)^2 + (v_0 t_f + y_0)^2 = |(x_0 v_0 - y_0 u_0)t_f + z_0|. \tag{21}$$

As can be easily seen from (21), for fixed u_0, v_0, the solution of this equation t_f tends to zero if x_0, y_0, z_0 tend simultaneously to zero. That proves (ii).

Simulations of the algorithm for two types of initial conditions are shown in Fig. 1.

The parameters $\alpha > 0, \beta$ define the size of the paraboloid. When $\beta/\alpha \to \infty$ the parabolic region $\mathscr{P}$ limits to the z-axis. From that point of view, to stabilize the system (1)–(3), it is reasonable to increase β as the state approaches the origin (if we decrease α the convergence of x and y will be slower). To realize this idea we can use a control law, where α increases when x and y approach the origin:

$$u = -\alpha x + \beta\frac{y}{x^2 + y^2}\,\mathrm{sign}(z), \tag{22}$$

$$v = -\alpha y - \beta\frac{x}{x^2 + y^2}\,\mathrm{sign}(z), \tag{23}$$

or even

$$u = -\alpha x + \beta\frac{y}{x^2 + y^2}\,z, \tag{24}$$

$$v = -\alpha y - \beta\frac{x}{x^2 + y^2}\,z. \tag{25}$$

As mentioned above for a detailed analysis in the case (24), (25), see Khennouf and Canudas de Wit [16].

Then from (3) we have

$$\dot{z} = -\beta\,\mathrm{sign}(z) \tag{26}$$

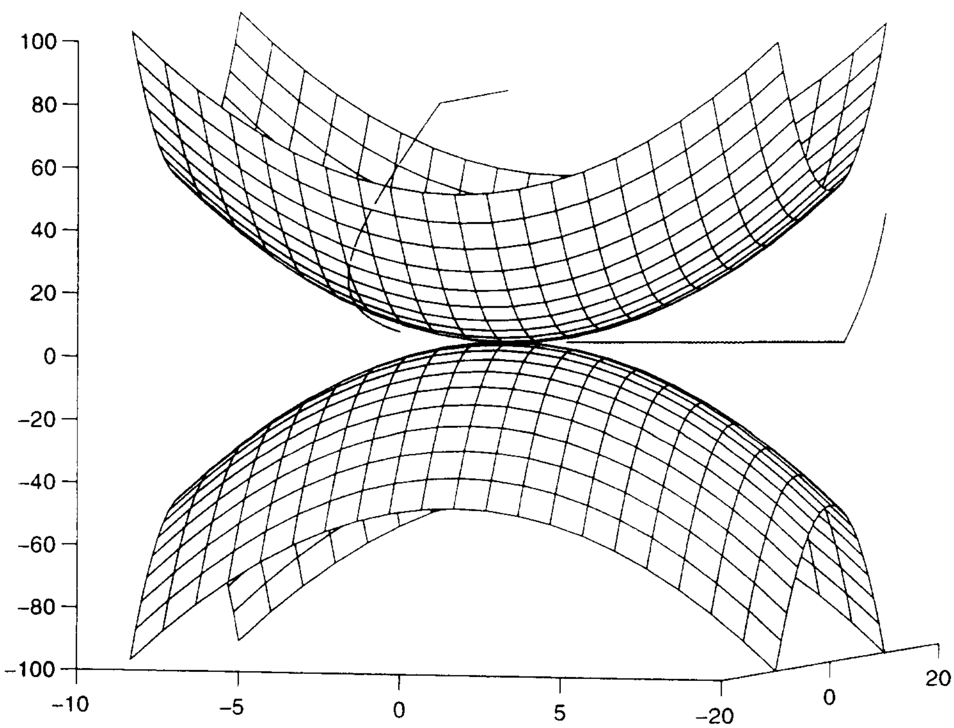

Fig. 1. Stabilization of the nonholonomic integrator.

or

$$\dot{z} = -\beta z, \tag{27}$$

respectively.

In both cases, the state converges to the origin from any initial conditions, except the ones belonging to the z-axis. But, in contrast to (4), (5) the control laws (22)–(25) are unbounded in the neighborhood of the z-axis (on the axis they are not defined). If the initial conditions belong to this set again we can apply any nonzero constant control for an arbitrary small period of time and then switch to (22), (23) or (24), (25). A method of dealing with the boundedness problem is also described by Khennouf and Canudas de Wit. Another global (excluding $\{x = 0\} \cap \{y = 0\}$ space), but only ε-stabilizing control, may be obtained by switching α:

Let α be the following function of x and y:

$$\alpha = \alpha_0 \operatorname{sign}(x^2 + y^2 - \varepsilon^2), \tag{28}$$

where $\alpha_0 > 0$, $\beta > 0$ are constants and let the control be

$$u = -\alpha x + \beta y z, \tag{29}$$

$$v = -\alpha y - \beta x z. \tag{30}$$

Using (7) we find that from any initial conditions x and y the state reaches an ε-sphere of the x, y-space origin:

$$x^2 + y^2 = \text{const} = \varepsilon^2. \tag{31}$$

After that the equation for the variable z in sliding mode is

$$\dot{z} = -\beta \varepsilon^2 z. \tag{32}$$

Therefore, $z \to 0$ when $t \to \infty$, while the variables x and y stay in an ε-vicinity of the origin. Of course, in (29), (30) z can be replaced by any function $g(z)$ which guarantees asymptotic stability of the equation

$$\dot{z} = -\beta \varepsilon^2 g(z), \tag{33}$$

for example, $g(z) = \operatorname{sign}(z)$.

Another interesting control can be obtained if in (28) we replace ε^2 by $|z|$:

$$\alpha = \begin{cases} \alpha_0 & \text{if } x^2 + y^2 > |z|, \\ \alpha_1 & \text{if } x^2 + y^2 \leqslant |z|, \end{cases} \tag{34}$$

where $\alpha_0 > 0$ and $\alpha_1 \leqslant 0$,

$$\alpha = \alpha_0 \operatorname{sign}(x^2 + y^2 - |z|) \tag{35}$$

and

$$u = -\alpha x + \beta y \operatorname{sign}(z), \tag{36}$$

$$v = -\alpha y - \beta x \operatorname{sign}(z). \tag{37}$$

In this case outside the parabolic region $x^2 + y^2 > |z|$ the asymptotic convergence of z is guaranteed. If the initial conditions are inside this region, $x^2 + y^2$ is increasing and reaches the parabola in finite time, remaining in sliding mode on the surface of parabola, where

$$\dot{z} = -\beta z. \tag{38}$$

In fact, this control forms two sliding surfaces in the state space of the closed system: $\{z = 0\}$ and $\{x^2 + y^2 = |z|\}$.

3. Tracking in the nonholonomic integrator

In this section we consider now the problem of tracking the trajectory $X^* = (x^*, y^*, z^*)^{\mathrm{T}}$:

$$x^* = x^*(t), \tag{39}$$

$$y^* = y^*(t), \tag{40}$$

$$z^* = z^*(t). \tag{41}$$

It will be shown that for any $\varepsilon > 0$ there exists a control in the form of feedback

$$u = U_\varepsilon(X, X^*), \tag{42}$$

$$v = V_\varepsilon(X, X^*), \tag{43}$$

so that the trajectory of the system reaches an ε-neighborhood of the desired reference curve in finite time t_1 and then stays there, i.e.

$$\rho(X, X^*) = (|x - x^*|^2 + |y - y^*|^2 + |z - z^*|^2)^{1/2} \leqslant \varepsilon \quad \text{for } t \geqslant t_1. \tag{44}$$

Further, the accuracy of tracking ε and the interception time t_1 can be chosen arbitrarily.

Let $\hat{z}$ be defined as

$$\hat{z}(t) = z(t) - x^*(t)y(t) + y^*(t)x(t). \tag{45}$$

Using (1)–(8) and (45) the derivative of $\hat{z}$ can be written as

$$\dot{\hat{z}} = (x - x^*)(v - \dot{y}^*) - (y - y^*)(u - \dot{x}^*) + g(t,x,y), \tag{46}$$

where

$$g(t,x,y) = 2x\dot{y}^* - \dot{y}^*x^* - 2y\dot{x}^* + \dot{x}^*y^*. \tag{47}$$

The problem of tracking x^*, y^*, z^* by the variables x, y, z is equivalent to one of stabilizing $\bar{x} = x - x^*$, $\bar{y} = y - y^*$, $\bar{z} = \hat{z} - z^*$.

The system (1)–(8) in the new variables can be written as

$$\dot{\bar{x}} = \bar{u}, \tag{48}$$

$$\dot{\bar{y}} = \bar{v}, \tag{49}$$

$$\dot{\bar{z}} = \bar{x}\bar{v} - \bar{y}\bar{u} + \bar{g}, \tag{50}$$

where we used the notations

$$\bar{u} = u - \dot{x}^*, \tag{51}$$

$$\bar{v} = v - \dot{y}^*, \tag{52}$$

$$\bar{g} = g - \dot{z}^*. \tag{53}$$

If we apply control of the type (4), (5),

$$\bar{u} = -\alpha\bar{x} + \beta\bar{y}\,\mathrm{sign}(\bar{z}), \tag{54}$$

$$\bar{v} = -\alpha\bar{y} - \beta\bar{x}\,\mathrm{sign}(\bar{z}), \tag{55}$$

to (48)–(50) we obtain

$$\dot{\bar{x}} = -\alpha\bar{x} + \beta\bar{y}\,\mathrm{sign}(\bar{z}), \tag{56}$$

$$\dot{\bar{y}} = -\alpha\bar{y} + \beta\bar{x}\,\mathrm{sign}(\bar{z}), \tag{57}$$

$$\dot{\bar{z}} = -\beta(\bar{x}^2 + \bar{y}^2)\,\mathrm{sign}(\bar{z}) + \bar{g}. \tag{58}$$

In similar fashion to (28) we assume that α is not a constant but the following function of $\bar{x}$ and $\bar{y}$:

$$\alpha = \begin{cases} \alpha_0 & \text{if } \bar{x}^2 + \bar{y}^2 > \varepsilon^2, \\ \alpha_1 & \text{if } \bar{x}^2 + \bar{y}^2 \leqslant \varepsilon^2, \end{cases} \tag{59}$$

where $\alpha_0 > 0$, $\alpha_1 \leqslant 0$ are constants. Let us consider a Lyapunov function

$$V = \bar{x}^2 + \bar{y}^2. \tag{60}$$

Its derivative along the trajectories of the system (56)–(58) is

$$\dot{V} = -2\alpha V. \tag{61}$$

Since $\alpha = \alpha(V)$ is zero if $V < \varepsilon^2$ and α_0 otherwise, the function V is decreasing until the ε-neighborhood of the origin ($\bar{x}, \bar{y}$-subspace) is reached, then $V = \mathrm{const} = \varepsilon^2$. After that moment Eq. (58) is

$$\dot{\bar{z}} = -\beta\varepsilon^2\,\mathrm{sign}(\bar{z}) + \bar{g}. \tag{62}$$

For sufficiently large β such that

$$\beta\varepsilon^2 > |\bar{g}| \tag{63}$$

in Eq. (62), sliding occurs, and

$$\bar{z} \equiv 0 \quad \text{for } t > t_1, \tag{64}$$

where the time t_1 can be chosen arbitrary close to the initial moment by increasing β.

In general, β should be chosen to be a function of x, y and $X^*, \dot{X}^*$ to satisfy (63), but as follows from (47) and (53), and due to the separate convergence of $\bar{x}, \bar{y}$ to the ε-neighborhood of the origin, for a bounded reference curve with bounded derivative β can be a constant. Thus, (44) holds, and we have proved the following theorem.

Theorem 3. *For any $\varepsilon > 0$ and $t_1 > 0$ there exists a positive constant $\alpha_0 > 0$, a positive function $\beta(X, X^*) > 0$, and a control $u = \bar{u} + \dot{x}^*$, $v = \bar{v} + \dot{y}^*$, where $\bar{u}$ and $\bar{v}$ are defined by expressions* (54), (55), (59), *such that the state trajectory $X(t) = (x(t), y(t), z(t))^{\mathrm{T}}$ enters an ε-neighborhood of the curve $X^* = (x(t), y(t), z(t))^{\mathrm{T}}$ not later than in time t_1 and stays in that neighborhood for all subsequent time.*

Let us note here that the tracking of the three-dimensional curve was achieved by a two-dimensional control.

4. Mechanical example: control of a knife edge

We illustrate the application of the above control strategy by an example studied also in the work of Bloch et al. [4]. The knife edge (or a skate) moving in the point contact on a plane surface is described by the equations

$$\ddot{x} = \lambda\sin\phi + u_1\cos\phi, \tag{65}$$

$$\ddot{y} = -\lambda\cos\phi + u_1\sin\phi, \tag{66}$$

$$\dot{\phi} = u_2, \tag{67}$$

where x, y and ϕ are the longitudinal and lateral positions of the point of contact and angular position with respect to the vertical axis, respectively. This reaction force λ can be excluded by using a nonholonomic constraint

$$\dot{x}\sin\phi - \dot{y}\cos\phi = 0. \tag{68}$$

Let us introduce the variables $v_x = \dot{x}$, $v_y = \dot{y}$, $v_\phi = \dot{\phi}$. Differentiating (68) and using (65)–(67) we obtain

$$\lambda = -(v_x\cos\phi + v_y\sin\phi)u_2. \tag{69}$$

Substituting (69) into (65)–(67) we obtain

$$\dot{v}_x = u_2(v_x\cos\phi + v_y\sin\phi)\sin\phi + u_1\cos\phi, \tag{70}$$

$$\dot{v}_y = -u_2(v_x\cos\phi + v_y\sin\phi)\cos\phi + u_1\sin\phi, \tag{71}$$

$$\dot{\phi} = u_2. \tag{72}$$

We consider the problem of stabilizing the lateral motion of the knife $v_y^*(t) \equiv 0$, $\phi^*(t) \equiv 0$, while tracking by the point of contact some desired longitudinal velocity $v_x^*(t)$.

According to Brockett [5] a nonholonomic system of the form $\dot{x} = B(x)u$ at least locally, can be written in a form similar to that of the nonholonomic integrator (since (70)–(72) is of third order, exactly like the nonholonomic integrator in our case).

Using Taylor expansion in the vicinity of $v_y = 0$, $\phi = 0$ the system (70)–(72) can be written as

$$\dot{v}_x = \bar{u}_1 + r_x, \tag{73}$$

$$\dot{\phi} = u_2, \tag{74}$$

$$\dot{v}_y = \phi\,\bar{u}_1 - v_x u_2 + r_y, \tag{75}$$

where $\bar{u}_1 = u_1 + u_2 v_x\phi$ and r_x and r_y contain terms of second order and higher and have vanishing first partials.

The control u_1 has the form

$$u_1 = \bar{u}_1 - u_2 v_x\phi, \tag{76}$$

where in accordance with the result from the previous section

$$\bar{u}_1 = \dot{v}_x^* - \alpha(v_x - v_x^*) + \beta\phi\,\mathrm{sign}(v_y - v_x^*\phi), \tag{77}$$

$$u_2 = -\alpha\phi - \beta(v_x - v_x^*)\,\mathrm{sign}(v_y - v_x^*\phi), \tag{78}$$

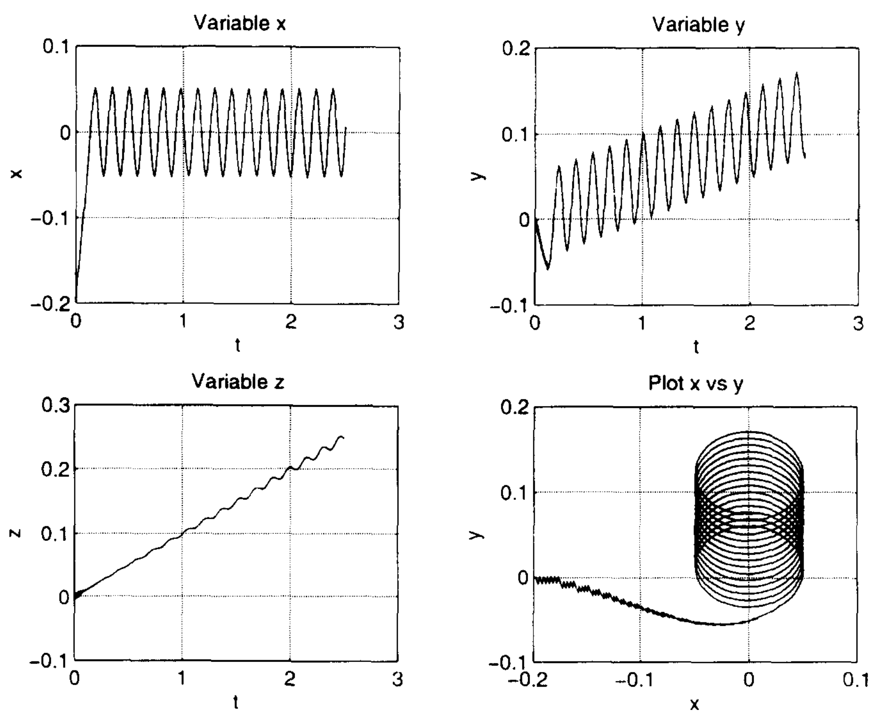

Fig. 2. Tracking in the nonholonomic integrator.

where

$$\alpha = \begin{cases} \alpha_0 & \text{if } (v_x - v_x^*)^2 + \phi^2 < \varepsilon^2, \\ 0 & \text{if } (v_x - v_x^*)^2 + \phi^2 \geqslant \varepsilon^2, \end{cases} \tag{79}$$

β and α_0 are positive constants.

Numerical example. We consider the tracking by the nonholonomic integrator of the following trajectory:

$$x^*(t) \equiv 0, \tag{80}$$

$$y^*(t) \equiv 0.05t, \tag{81}$$

$$z^*(t) \equiv 0.1t. \tag{82}$$

The parameters of the algorithm are the following:

$$\varepsilon = 0.05, \tag{83}$$

$$\alpha_0 = 10, \tag{84}$$

$$\beta = 40. \tag{85}$$

The simulation plots for initial conditions $x_0 = -0.2$, $y_0 = 0$, $z_0 = 0$ are shown in Fig. 2. As can be seen the variables x and y oscillate within the prescribed ε-vicinity of the origin (rotating on the x, y-plane) as the variable z tracks the desired function.

Remark 1. The accuracy of the tracking ε can be chosen arbitrarily, but smaller values lead to larger control.

Remark 2. The designed control provides global convergence to an ε-vicinity of the desired trajectory in any prescribed finite time but, again, the cost of small convergence time is an increase in control magnitude.

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