Robust Nonlinear Motion Control of a Helicopter

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***Abstract*—We consider the problem of controlling the vertical motion of a nonlinear model of a helicopter, while stabilizing the lateral and horizontal position and maintaining a constant attitude. The reference to be tracked is given by a sum of a constant and a fixed number of sinusoidal signals, and it is assumed not to be available to the controller. This represents a possible situation in which the controller is required to synchronize the vehicle motion with that of an oscillating platform, such as the deck of a ship in high seas. We design a nonlinear controller which combines recent results on nonlinear adaptive output regulations and robust stabilization of systems in feedforward form by means of saturated controls. Simulation results show the effectiveness of the method and its ability to cope with uncertainties on the plant and actuator model.**

***Index Terms*—Nonlinear systems, output regulation, robust control, saturated controls, vertical takeoff and landing.**

I. Introduction

AUTOPILOT design for helicopters is a challenging testbed in nonlinear feedback design, due to the nonlinearity of the dynamics and the strong coupling between the forces and torques produced by the vehicle actuators, as witnessed by a good deal of important contributions in the last twenty years (see [1]–[8] to mention a few). A helicopter is, in general, an underactuated mechanical system, that is, a system possessing more degrees of freedom than independent control inputs. Partial (i.e, input-output) feedback linearization techniques are not suitable for the control of such a system, because the resulting zero-dynamics are only critically stable. Moreover, the model may be affected by large uncertainties and unmodeled dynamics, and this also renders any design technique based on exact cancellation of nonlinear terms poorly suited. In this paper, we address the design of an internal-model based autopilot for a helicopter. The control goal is to have the vertical position of the helicopter tracking an exogenous reference trajectory, while its longitudinal and lateral position, as well as its attitude, are stabilized to a constant configuration. The reference trajectory which is to be tracked is a superposition of a finite number of sinusoidal signals of *unknown* frequency, amplitude and phase. This situation corresponds, for instance, to the case in which a helicopter is required to land *autonomously* on the deck of a ship subject to wave-induced oscillations. The trajectory in question is not available in real time: rather only the tracking error and its rate of change are assumed to be available in real time. A similar problem has been previously considered and solved for a simplified model of a VTOL aircraft [9]. With respect to the former, however, the present case is more challenging, due to the higher complexity of the vehicle dynamics which renders the stabilization onto the desired trajectory a difficult task. We propose a solution which combines recent results on nonlinear adaptive regulation and robust stabilization of systems in feedforward form by means of saturated controls. The focus of this paper is mostly on the stabilization technique. Due to the intrinsic robustness of the method, we expect the controller to perform satisfactorily despite the effect of parametric uncertainties and unmodeled dynamics. As a matter of fact, we design our controller on the basis of a simplified model, and show the effectiveness of our method on a more complete model by means of computer simulations. Complete model and simplified model are precisely those proposed in [2]. Our design techniques assume full availability of all state variables in appropriate reference frames; namely vertical, longitudinal, lateral errors (and their rates of change) as well as attitude (and its rate of change). This makes it possible to develop a semiglobal robust stabilization scheme, thus circumventing the problem that, for certain selections of output variables, the controlled system is nonminumum phase (as shown in [2]). The paper is organized as follows: in Section II the vehicle model is introduced. In Section III we describe the design problem, and in Sections IV and V we present the controller design. Simulation results are illustrated and briefly discussed in Section VI. Finally, we draw some conclusions in Section VII.

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II. Helicopter Model

A mathematical model of the helicopter dynamics can be derived from Newton–Euler equations of motion of a rigid body in the configuration space $SE(3) = \mathbb{R}^3 \times SO(3)$. Fix an inertial coordinate frame $\mathcal{F}^i$ in the euclidean space, and fix a coordinate frame $\mathcal{F}^b$ attached to the body. Let $p^i = \mathrm{col}(x, y, z) \in \mathbb{R}^3$ denote the position of the center of mass of the rigid body with respect to the origin of $\mathcal{F}^i$, and let $R \in SO(3)$ denote the rotation matrix mapping vectors expressed in $\mathcal{F}^b$ coordinates into vectors expressed in $\mathcal{F}^i$ coordinates. The translational velocity $v^b \in \mathbb{R}^3$ of the center of mass of the body and its angular velocity $\omega^b \in \mathbb{R}^3$ (both expressed in $\mathcal{F}^b$) by definition satisfy

$$\begin{aligned} \dot{p}^i &= Rv^b \\ \dot{R} &= RS(\omega^b) \end{aligned} \qquad (1)$$

where

$$S(x) = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}.$$

Moreover

$$\begin{aligned} M\dot{v}^b &= -MS(\omega^b)v^b + f^b \\ J\dot{\omega}^b &= -S(\omega^b)J\omega^b + \tau^b \end{aligned} \tag{2}$$

where $F^b = (f^b, \tau^b)$ is the external wrench in body-fixed coordinates and M and J the mass and the inertia tensor of the body. We will parametrize the group of rotation matrices by means of unit quaternions $\mathbf{q} = (q_0, q) \in \mathbb{R}^4$, where q_0 and $q = (q_1, q_2, q_3)^T$ denote respectively the scalar and the vector parts of the quaternion, satisfying the constraint

$$q_0^2 + \|q\|_2^2 = 1.$$

Accordingly, the rotation matrix is given as

$$R = \begin{pmatrix} 1-2q_2^2-2q_3^2 & 2q_1q_2-2q_0q_3 & 2q_1q_3+2q_0q_2 \\ 2q_1q_2+2q_0q_3 & 1-2q_1^2-2q_3^2 & 2q_2q_3-2q_0q_1 \\ 2q_1q_3-2q_0q_2 & 2q_2q_3+2q_0q_1 & 1-2q_1^2-2q_2^2 \end{pmatrix}. \tag{3}$$

The second equation in (1) is then replaced by the quaternion propagation equation

$$\begin{aligned} \dot{q}_0 &= -\frac{1}{2}q^T\omega^b \\ \dot{q} &= \frac{1}{2}[q_0 I + S(q)]\omega^b \end{aligned}$$

while the motion of the center of mass of the rigid body is expressed in inertial coordinates as

$$M\ddot{p}^i = Rf^b. \tag{4}$$

In the specific case of a helicopter, the wrench (f^b, τ^b) is provided by the forces and torques generated by the rotors and the aerodynamic forces. Following [2], the thrusts generated by the main rotor and the tail rotor are denoted by T_M and T_T, respectively. The main rotor shaft is directed along the body z^b axis, while the tip path plane of the main rotor is tilted by an angle a around the y^b axis and by an angle b around the x^b axis. The overall control input is provided by the vector (T_M, T_T, a, b).

The expressions of f^b and τ^b in terms of the four components of the control vector (T_M, T_T, a, b), the mechanical parameters and the aerodynamic coefficients can be found, for instance, in [2]. In particular, following [2], since the tilt angles a and b are small, we let

$$\begin{aligned} \sin(a) &\approx a \quad \sin(b) \approx b \\ \cos(a) &\approx 1 \quad \cos(b) \approx 1. \end{aligned} \tag{5}$$

Also, we neglect the contribution of T_M along the x^b direction and we assume that the contribution of T_T and along the y^b direction is matched by that of T_M, thus obtaining the following simplified model for f^b:

$$f^b = \begin{pmatrix} 0 \\ 0 \\ -T_M \end{pmatrix} + R^T \begin{pmatrix} 0 \\ 0 \\ Mg \end{pmatrix}. \tag{6}$$

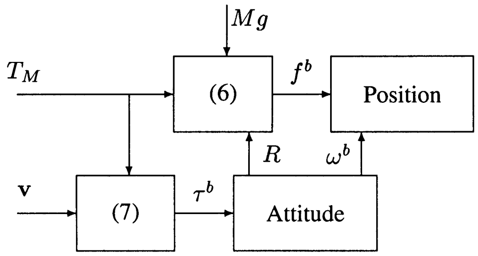

Fig. 1. Model of the approximated system dynamics.

As far as the external torque τ^b is concerned, under the previous hypotheses, it is seen from [2] that

$$\tau^b(\mathbf{v}) = A(T_M)\mathbf{v} + B(T_M) \quad \mathbf{v} := \text{col}(a, b, T_T) \tag{7}$$

in which $A(T_M)$ and $B(T_M)$ are, respectively, a matrix and a vector of affine functions of the thrust T_M, whose coefficients depend on the geometry of the helicopter and on the coefficients which characterize the aerodynamic forces. A sketch of the position/attitude dynamics is reported in Fig. 1.

One of the goals of the paper is to design a controller able to deal with possibly large parameter uncertainties, including the mass M of the vehicle, its inertia tensor J, and the aerodynamic coefficients in (7). Collecting all possible parameters subject to uncertainty in a single vector μ, we let μ_0 stand for its nominal value and $\mu_\Delta = \mu - \mu_0$ for the additive uncertainty. It is assumed that $\mu_\Delta \in \mathcal{P}$, a given compact set. Accordingly, we set

$$M = M_0 + M_\Delta \quad J = J_0 + J_\Delta$$

and, bearing in mind the fact that $A(T_M)$ and $B(T_M)$ are functions of T_M

$$\begin{aligned} A(T_M) &= A_0(T_M) + A_\Delta(T_M) \\ B(T_M) &= B_0(T_M) + B_\Delta(T_M) \end{aligned} \tag{8}$$

with obvious meaning of the subscripts.

III. Problem Statement

The goal of this paper is the design of an autopilot able to secure smooth landing of the helicopter on an oscillating platform in uncertain conditions. The considered setup represents a possible scenario in which a helicopter is required to perform a smooth landing on a deck of a ship which, due to wave motion, is subject to large vertical oscillation. The control objective can be conveniently divided into two separate tasks: the first is the synchronization of the vertical motion of the helicopter with that of the deck at a given distance H. Once synchronization has been achieved, the second task is to provide a smooth landing, letting the vertical offset H decay to zero. Clearly, the crucial part is the design of a controller to accomplish the first task. The problem becomes quite challenging if the information available for feedback is provided by passive sensors only, yielding the relative position between the helicopter and the deck. If this is the case, the vertical reference trajectory to be tracked by the helicopter is not available, but must be estimated in real time by processing the synchronization error. This trajectory, denoted in what follows by $z^*(t)$, is modeled as the sum of a fixed number of sinusoidal signals of *unknown* amplitude, phase and frequency, namely as

$$z^*(t) = \sum_{i=1}^{N} A_i \cos(\Omega_i t + \varphi_i). \tag{9}$$

In this setting, the uncertainty on the reference trajectory consists in uncertainty on the exact value of the parameters $(A_i, \varphi_i, \Omega_i)$, $i = 1, \ldots, N$. Consequently, one of the main goals to be accomplished in the design is to let the center of mass of the helicopter asymptotically track, as accurately as possible, the reference motion

$$\left(x^{\text{ref}}(t), y^{\text{ref}}(t), z^{\text{ref}}(t)\right) = (0, 0, H + z^*(t)). \tag{10}$$

It is also appropriate to require that the vehicle's attitude asymptotically tracks, as accurately as possible, the constant reference $R^{\text{ref}}(t) = I$, which corresponds to the following possible choice for the quaternion[1]

$$q_0^{\text{ref}}(t) = 1 \quad q^{\text{ref}}(t) = (0, 0, 0)^T. \tag{11}$$

The problem of having $z(t)$ to track $z^{\text{ref}}(t)$ can be naturally cast in the framework of *nonlinear adaptive output regulation* theory (see [11], [12]), as the signal $z^{\text{ref}}(t)$ is generated by a linear time-invariant *exosystem*

$$\begin{aligned} z^{\text{ref}}(t) =& H + r(w) \\ \dot{H} =& 0 \\ \dot{w} =& S(\varrho) w \end{aligned}$$

in which $\varrho = \text{col}(\Omega_1, \ldots, \Omega_N)$

$$S(\varrho) = \text{diag}(S_1, \ldots, S_N)$$

with

$$S_i = \begin{pmatrix} 0 & \Omega_i \\ -\Omega_i & 0 \end{pmatrix}, \qquad i = 1, \ldots, N$$

and $r(w) = Qw$, with Q defined in an obvious way. As customary, we assume that the values of ϱ range over a given compact set. Note that the role of the parameters $(A_1, \varphi_1), \ldots, (A_N, \varphi_N)$ of (9) is played by the initial condition $w(0)$ of the exosystem. As far as the tracking goal for $x(t)$, $y(t)$, and $q(t)$ is concerned, we seek to obtain *ultimate boundedness* by *arbitrarily small bounds*. Setting

$$\mathbf{e} := \begin{pmatrix} x & y & z - z^{\text{ref}} \end{pmatrix} \quad \dot{\mathbf{e}} := \begin{pmatrix} \dot{x} & \dot{y} & \dot{z} - \dot{r} \end{pmatrix}$$

the design problem can be cast as follows: given any (arbitrarily small) number $\delta > 0$, design a smooth dynamic controller of the form

$$\begin{aligned} \dot{\eta} =& \varphi(\eta, \mathbf{e}, \dot{\mathbf{e}}, q, \omega^b) \\ T_M =& \psi_{T_M}(\eta, \mathbf{e}, \dot{\mathbf{e}}, q, \omega^b) \\ \mathbf{v} =& \psi_{\mathbf{v}}(\eta, \mathbf{e}, \dot{\mathbf{e}}, q, \omega^b) \end{aligned}$$

such that the tracking objectives

$$\begin{aligned} \lim_{t\to\infty} |z(t) - z^{\text{ref}}(t)| =& 0 \\ \|\mathbf{e}(t)\| \leq& \delta \text{ and } \|q(t)\| \leq \delta \text{ for all } t \geq T \end{aligned}$$

are attained within a semiglobal domain of attraction (that is, from initial conditions for the plant states in an arbitrarily large compact set), for all admissible values of the parameters of the plant and the exosystem. It is worth noting that the controller is allowed to process the tracking error $\mathbf{e}$ of the center of mass and its derivative $\dot{\mathbf{e}}$, *but not* the state $w(t)$ of the exosystem and the vertical position $z(t)$. Finally, note that the steady-state value of the main thrust T_M needed to keep the helicopter on the reference trajectory (10) and (11) is given by $T_M^*(t) = M(g - \ddot{r}(w(t)))$. Since we require T_M to be positive, we must have

$$|\ddot{r}(w(t))| < g \qquad \forall\, t \geq 0 \tag{12}$$

which gives an upper bound to the admissible initial conditions of the exosystem.

IV. Stabilization of the Vertical Error Dynamics

The first step in the regulator design is the computation of the feedforward control signal that must be imposed to achieve zero error in steady state. In the terminology of output regulation theory, this amounts in solving the *regulator equations* for the problem under investigation (see [10], [11]). To this end, consider the equation for $z(t)$, readily obtained from (1), (6), and (3)

$$M\ddot{z} = -\left(1 - 2q_1^2 - 2q_2^2\right) T_M + Mg.$$

To compensate for the nominal value of the gravity force, let us choose the preliminary control law

$$T_M = \frac{gM_0 - u}{1 - \text{sat}_c(2q_1^2 + 2q_2^2)} \tag{13}$$

where $0 < c < 1$, the function $\text{sat}_c(s)$ is the standard saturation function

$$\text{sat}_c(s) := \text{sgn}(s) \min\{|s|, c\}$$

and u is an additional control to be defined. The equation for the vertical dynamics is described by

$$M\ddot{z} = \phi_c^z(q) u + g\left[M - M_0 \phi_c^z(q)\right] \tag{14}$$

where[2]

$$\phi_c^z(q) = \frac{1 - 2q_1^2 - 2q_2^2}{1 - \text{sat}_c\left(2q_1^2 + 2q_2^2\right)}.$$

From this, it is concluded that, if $q(t)$ is small so that $\phi_c^z(q(t)) \equiv 1$, the input u needed to keep $z(t) \equiv z^{\text{ref}}(t)$ is simply (recall that $\ddot{z}^{\text{ref}} = \ddot{r}(w)$)

$$u_{\text{ss}} = M\ddot{r}(w) - gM_\Delta = MQS^2(\varrho)w - gM_\Delta. \tag{15}$$

The steady-state behavior u_{ss} of u is the superposition of a term meant to enforce the vertical reference acceleration and a term meant to compensate the residual gravity force.

[1]It is worth stressing that this desired attitude configuration is compatible with the steady state requirement (10) because we are assuming the simplified model (6) for the force generation. In the general case, assuming the force generation model as presented in [2], the desired motion (10) is achieved with a steady state attitude motion different from (11), as described in [10].

[2]Note that $2q_1^2 + 2q_2^2 \leq c$ implies $\phi_c^z(q) = 1$.

It is clear that, as $u_{\rm ss}$ depends on unknown parameters and on the unmeasurable state w, the steady-state control (15) *is not* directly implementable as a "feedforward" control. However, it can be asymptotically reproduced by means of a linear internal model, as $r(w)$ is a linear function of w. Accordingly, we choose the control u as the sum of a stabilizing control and the output of an internal model, i.e., $u = u_{\rm st} + u_{\rm im}$. The internal model will be designed on the basis of *adaptive output regulation theory* (see [9] and [12]). In fact, (14) is a two-dimensional system having relative degree 2. Hence, the hypotheses presented in [12] for the design of an adaptive internal model, hold. Define the observable pair $(\Phi(\varrho), \Gamma(\varrho))$ as

$$\Phi(\varrho) = \begin{pmatrix} 0 & 0 \\ 0 & S(\varrho) \end{pmatrix} \qquad \Gamma(\varrho) = \begin{pmatrix} 1 & \Gamma_2(\varrho) \end{pmatrix}$$

in which

$$\Gamma_2(\varrho) = \begin{pmatrix} -\Omega_1^2 & 0 & -\Omega_2^2 & 0 & \dots & -\Omega_N^2 & 0 \end{pmatrix}$$

and note that, by construction, the map

$$\tau(w,\mu) = \begin{pmatrix} -gM_\Delta \\ Mw \end{pmatrix}$$

satisfies, for every μ and ϱ, the *immersion condition*

$$\begin{aligned} \frac{\partial \tau}{\partial w} S(\varrho) w =& \Phi(\varrho)\tau(w,\mu) \\ u_{\rm ss} =& \Gamma(\varrho)\tau(w,\mu). \end{aligned} \tag{16}$$

If the vector ϱ was known precisely, the matrices $\Phi(\varrho)$ and $\Gamma(\varrho)$ could be directly used for the design of the internal model. Conversely, if ϱ is not known, a further step is needed. Let F_2 be a $2N \times 2N$ Hurwitz matrix and G_2 be $2N \times 1$ vector such that the pair (F_2, G_2) is controllable. Then, using standard passivity arguments, it is easy to show that there always exists a $1 \times 2N$ matrix H_2 such that the pair

$$F = \begin{pmatrix} 0 & H_2 \\ -G_2 & F_2 \end{pmatrix} \quad G = \begin{pmatrix} 0 \\ G_2 \end{pmatrix} \tag{17}$$

is controllable, and the matrix F is Hurwitz. From [12], it is known that, for any vector $\varrho \in \mathbb{R}^N$, there exists a $1 \times (1 + 2N)$ row vector Ψ_ϱ, of the form

$$\Psi_\varrho = \begin{pmatrix} 1 & \Psi_{2,\varrho} \end{pmatrix}$$

such that the pair $(\Phi(\varrho), \Gamma(\varrho))$ is *similar* to the pair $(F + G\Psi_\varrho, \Psi_\varrho)$. As a consequence, there exists a map $\bar\tau_\varrho(w,\mu)$ that satisfies, for every μ and ϱ, the immersion condition

$$\begin{aligned} \frac{\partial \bar\tau_\varrho}{\partial w} S(\varrho) w =& (F + G\Psi_\varrho)\bar\tau_\varrho(w,\mu) \\ u_{\rm ss} =& \Psi_\varrho \bar\tau_\varrho(w,\mu). \end{aligned} \tag{18}$$

Denote now by e_z and $\dot e_z$ the third components of the vectors $\mathbf{e}$ and $\dot{\mathbf{e}}$, namely

$$e_z := z - z^{\rm ref} \qquad \dot e_z = \dot z - \dot r$$

and consider, as an internal model for our problem, the system

$$\begin{aligned} \begin{pmatrix} \dot\xi_1 \\ \dot\xi_2 \end{pmatrix} =& \begin{pmatrix} 0 & H_2 \\ 0 & F_2 + G_2\hat\Psi_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + N(u_{\rm st}, e_z) \\ u_{\rm im} =& \xi_1 + \hat\Psi_2 \xi_2 \end{aligned}$$

where

$$N(u_{\rm st}, e_z) = \begin{pmatrix} 0 \\ G_2 \end{pmatrix} u_{\rm st} - \begin{pmatrix} H_2 \\ F_2 \end{pmatrix} G_2 M_0 \dot e_z$$

in which $\xi_1 \in \mathbb{R}$, $\xi_2 \in \mathbb{R}^{2N}$ and $\hat\Psi_2$ is a $1 \times 2N$ row vector. The control system is rewritten in the form

$$\begin{aligned} \dot\xi =& (F + G\hat\Psi)\xi + Gu_{\rm st} - FGM_0\dot e_z \\ u =& \hat\Psi\xi + u_{\rm st} \end{aligned} \tag{19}$$

with $\hat\Psi = \begin{pmatrix} 1 & \hat\Psi_2 \end{pmatrix}$. In case the vector ϱ is known, we set $\hat\Psi_2 = \Psi_{2,\varrho}$. Otherwise, we consider $\hat\Psi_2$ to be a vector of parameter estimates to be adapted, and we choose the update law (see [12])

$$\dot{\hat\Psi}_2 = -\gamma \xi_2^T (\dot e_z + k_1 e_z) \tag{20}$$

with k_1 and γ positive design parameters. The control law u is then completed choosing the *high-gain* stabilizing feedback

$$u_{\rm st} = -k_2(\dot e_z + k_1 e_z) \tag{21}$$

where $k_2 > 0$ is a design parameter. Changing coordinates as

$$\begin{aligned} \xi \mapsto \chi :=& \xi - \bar\tau_\varrho(w,\mu) - GM\dot e_z \\ \hat\Psi_2 \mapsto \tilde\Psi_2 :=& \hat\Psi_2 - \Psi_{2,\varrho} \end{aligned} \tag{22}$$

and letting $\tilde\Psi = \begin{pmatrix} 0 & \tilde\Psi_2 \end{pmatrix}$, the e_z, ξ, $\hat\Psi_2$ dynamics in the new coordinates read as

$$\begin{aligned} M\ddot e_z =& u_{\rm st} + \Psi_\varrho \chi + \Psi_\varrho GM\dot e_z + \tilde\Psi_2 \xi_2 + \upsilon \\ \dot\chi =& F\chi + FGM_\Delta \dot e_z + G\upsilon \\ \dot{\tilde\Psi}_2 =& -\gamma\xi_2^T(\dot e_z + k_1 e_z). \end{aligned} \tag{23}$$

where $\upsilon = (1 - \phi_c^z(q))(gM_0 - u)$. This system, setting

$$\mathbf{z} := \begin{pmatrix} e_z & \dot e_z & \chi & \tilde\Psi_2 \end{pmatrix}^T \tag{24}$$

can be rewritten in the form

$$\dot{\mathbf{z}} = f(\mathbf{z}, t) + g(\mathbf{z}, t)(1 - \phi_c^z(q)). \tag{25}$$

Note that the dependence on t of the vector fields $f(\mathbf{z}, t)$ and $g(\mathbf{z}, t)$ arises from the dependence on t of the term $\bar\tau_\varrho(w,\mu)$ in $\xi(t)$, in turn induced by the dependence on t of w. Following [9] and [12], it can be shown that, for a sufficiently large k_2, system

$$\dot{\mathbf{z}} = f(\mathbf{z}, t) \tag{26}$$

(or, what is the same, system (23), *if* $\|q\|$ is sufficiently small so that $\phi_c^z(q) = 1$) has a globally asymptotically stable equilibrium at $(e_z, \dot e_z, \chi, \tilde\Psi_2) = (0, 0, 0, \tilde\Psi_2^\star)$ for some $\tilde\Psi_2^\star$ which, in turn, coincides with the origin in case all the modes of the exosystem are excited. Without loss of generality, we henceforth assume that this is the case.

This result concludes the stabilization of the vertical dynamics. In particular, in case the attitude is kept sufficiently close to the desired one so that $\phi_c^z(q) = 1$, the $(4N+1)$-order controller (13), (19) and (21) with adaptation law (20) is able to steer asymptotically the vertical error e_z to zero. In Sections V and VI we will show how to design a control law for the input $\mathbf{v}$ to simultaneously achieve the condition $\phi_c^z(q) = 1$ in finite time, and stabilize the lateral and longitudinal dynamics.

V. Design of the Stabilizer

A. Lateral and Longitudinal Dynamics

We start by deriving the expression of the lateral and longitudinal dynamics resulting from the choice of the main thrust T_M performed in Section IV. First, note that the term $gM_0 - u = gM_0 - u_{\text{im}} - u_{\text{st}}$ reads as

$$gM_0 - u = gM_0 - \hat{\Psi}\xi + k_2(\dot{e}_z + k_1 e_z)$$

which, keeping in mind the definition of χ in (22) and (18), yields

$$gM_0 - u = gM - M\ddot{r}(w) - y_{\mathbf{z}}(\mathbf{z}, w) \tag{27}$$

where

$$\begin{aligned} y_z(\mathbf{z}, w) = \tilde{\Psi}\bar{\tau}_\varrho(w,\mu) + (\tilde{\Psi} + \Psi_\varrho)(\chi + GM\dot{e}_z) \\ -k_2(\dot{e}_z + k_1 e_z). \end{aligned} \tag{28}$$

Note that, for all $w \in \mathbb{R}^{2N}$

$$y_z(\mathbf{0}, w) = 0. \tag{29}$$

Bearing in mind (4), (6), we obtain the following expression for the longitudinal dynamics:

$$\begin{aligned} \dot{x} =& x_2 \\ M\dot{x}_2 =& -\tilde{d}(t)q_2 + m(\mathbf{q},t)q_1q_3 + n_x(\mathbf{q})y_{\mathbf{z}}(\mathbf{z},w) \end{aligned} \tag{30}$$

where

$$\tilde{d}(t) = \frac{2(gM - M\ddot{r}(w(t)))q_0(t)}{1 - \text{sat}_c\,(2q_1^2(t) + 2q_2^2(t))} \tag{31}$$

$$\begin{aligned} m(\mathbf{q},t) =& -\frac{2(gM - M\ddot{r}(w(t)))}{1 - \text{sat}_c(2q_1^2 + 2q_2^2)} \\ n_x(\mathbf{q}) =& \frac{2q_1q_3 + 2q_0q_2}{1 - \text{sat}_c\,(2q_1^2 + 2q_2^2)}. \end{aligned} \tag{32}$$

Note that we have treated the presence of the forcing term $w(t)$ as a time-varying entry, while $\tilde{d}(t)$ plays the role of a bounded time-varying coefficient. Likewise, the lateral dynamics can be put in the form

$$\begin{aligned} \dot{y} =& y_2 \\ M\dot{y}_2 =& \tilde{d}(t)q_1 + m(\mathbf{q},t)q_2q_3 + n_y(\mathbf{q})y_{\mathbf{z}}(\mathbf{z},w) \end{aligned} \tag{33}$$

where

$$n_y(\mathbf{q}) = \frac{2q_2q_3 - 2q_0q_1}{1 - \text{sat}_c(2q_1^2 + 2q_2^2)}. \tag{34}$$

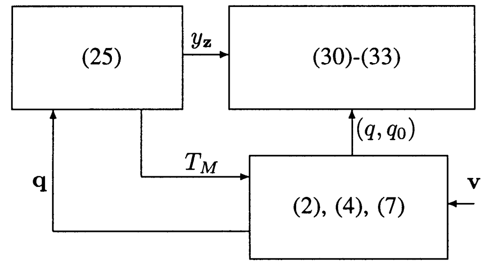

Fig. 2. Overall approximated system dynamics.

The $x-y$ dynamics are viewed as a system interconnected to the attitude and vertical dynamics, according to the structure depicted in Fig. 2. Basically, the choice of the control input $\mathbf{v}$ able to stabilize the overall system will rely upon the following considerations. We look at the $x-y$ subsystem as a system with "virtual control" q and exogenous input $y_{\mathbf{z}}$. The latter, according to the results presented in Section IV and by virtue of (28) and (29), is an asymptotically vanishing signal, *provided that the attitude variable* q is kept sufficiently smallby means of the control input $\mathbf{v}$. In the light of this, the control law $\mathbf{v}$ will be designed on one hand to force q to assume sufficiently small values so that $\phi_c^z(q) = 1$ in finite time and, on the other hand, to render the $x-y$ subsystem *input-to-state stable (ISS)* with respect to the input $y_{\mathbf{z}}$. According to classical results about input to state stability, this will provide asymptotic stability of the lateral and longitudinal dynamics. This task will be accomplished using a *partially saturated* control law, obtained combining a high gain controller for the attitude dynamics and a *nested saturation* controller for the $x-y$ dynamics. As it will be clarified in Section VI, the presence of the saturation function plays a crucial role in "decoupling" the attitude from the $x-y$ dynamics, in such a way that the two actions can be performed simultaneously.

B. Stabilization of the Attitude Dynamics

In this section, we deal with the problem of achieving the condition $\phi_c^z(q) = 1$ in finite time by a proper design of $\mathbf{v}$. First of all, we use a preliminary control law which is meant to remove the nominal part of $B(T_M)$ from (7) and (8), i.e., we choose

$$\mathbf{v} = A_0(T_M)^{-1}[\tilde{\mathbf{v}} - B_0(T_M)] \tag{35}$$

in which $\tilde{\mathbf{v}}$ is an additional control input to be defined. This yields a new expression for τ^b

$$\tau^b(\tilde{\mathbf{v}}) = L(T_M)\tilde{\mathbf{v}} + \Delta(T_M) \tag{36}$$

where

$$\begin{aligned} L(T_M) =& I + Z(T_M) \\ \Delta(T_M) =& B_\Delta(T_M) - Z(T_M)B_0(T_M). \end{aligned}$$

with $Z(T_M) := A_\Delta(T_M)A_0^{-1}(T_M)$. The control law $\tilde{\mathbf{v}}$ is then chosen as

$$\tilde{\mathbf{v}} = -K_4\omega^b - K_4K_3q + K_4K_3u_2 \tag{37}$$

where $K_3 > 0$ and $K_4 > 0$ are design parameters, and u_2 is an additional control input which is assumed to be bounded by a positive number λ_2, i.e.,

$$\|u_2(t)\| \le \lambda_2 \qquad \text{for all } t \ge 0. \tag{38}$$

The bound (38) will be enforced by choosing u_2 as a saturated function of the $x - y$ states. As it has already been remarked, the first goal of the control law (37) is to achieve the condition $\phi_c^z(q) = 1$ in finite time. To this end, we show that this can be accomplished by a suitable tuning of the design parameters K_3, K_4 and λ_2. However, since the expression of the torque τ^b in (36) depends on T_M which, according to (13) and (27), is a function of $\mathbf{z}$ and w, we first need to establish a result which guarantees boundedness of T_M. Fix an arbitrary compact set $\mathcal{Z}$ of initial conditions for $\mathbf{z}$ and let the initial condition for w range in a compact set $\mathcal{W}$ such that (12) holds for each trajectory $w(t)$ originating in $\mathcal{W}$. Pick $(\mathbf{z}_0, w_0) \in \mathcal{Z} \times \mathcal{W}$, and let $\bar{\mathbf{z}}_f(t)$ denote the corresponding integral curve of (26), which is known to asymptotically decay to 0 as $t \to \infty$. Then, there exists finite numbers T_M^l and T_M^u such that (for compactness, we drop the time t)

$$T_M^l \le \frac{-M(\ddot{r}(w) + g) - y_{\mathbf{z}}(\bar{\mathbf{z}}_f, w)}{1 - \mathrm{sat}_c\,(2q_1^2 + 2q_2^2)} \le T_M^u \tag{39}$$

for all $(\mathbf{z}_0, w_0) \in \mathcal{Z} \times \mathcal{W}$, for all q satisfying $\|q\| \le 1$ and for all $t \ge 0$. These bounds on T_M are instrumental in showing that for any arbitrary $T^\star$ a suitable choice of K_3, K_4 and λ_2 renders the condition $\phi_c^z(q(t)) = 1$ fulfilled for all $t \ge T^\star$. This is established in the next proposition.

Proposition 5.1: Suppose there exists $l_1^\star > 0$ such that[3]

$$0 < 2l_1^\star I \le L(T_M) + L^T(T_M) \quad \forall\, T_M \in [T_M^l, T_M^u]$$

and let $l_2^\star$, $\delta^\star > 0$ satisfy

$$\|L(T_M)\| \le l_2^\star \|\Delta(T_M)\| \le \delta^\star \quad \forall\, T_M \in [T_M^l, T_M^u].$$

Choose $0 < \varepsilon < 1$ arbitrarily, and fix compact sets $\mathcal{Q}$, Ω of initial conditions for $q(t)$ and $\omega^b(t)$, respectively, with $\mathcal{Q}$ contained in the set

$$\left\{ q \in \mathbb{R}^3 : \|q\| < \sqrt{1 - \varepsilon^2} \right\}.$$

Then, for any $T^\star > 0$ there exist a number $K_3^\star > 0$ and positive numbers $\lambda_2^\star(K_3)$ and $K_4^\star(K_3)$, both depending on K_3, such that for all $K_3 \ge K_3^\star$, $\lambda_2 \le \lambda_2^\star(K_3)$ and $K_4 \ge K_4^\star(K_3)$, the following hold.

a) The trajectories of the system

$$\begin{aligned} \dot{q}_0 &= -\frac{1}{2} q^T \omega^b \\ \dot{q} &= \frac{1}{2}[q_0 I + S(q)]\omega^b \\ J\dot{\omega}^b &= -S(\omega^b)J\omega^b + L(T_M)\tilde{\mathbf{v}} + \Delta(T_M) \\ \tilde{\mathbf{v}} &= -K_4\omega^b - K_4K_3q + K_4K_3u_2 \end{aligned} \tag{40}$$

with initial conditions $(q(0), \omega^b(0)) \in \mathcal{Q} \times \Omega$ and $q_0(0) > 0$ are bounded, and satisfy

$$q_0(t) > \varepsilon \qquad \forall\, t \ge 0.$$

b) $\phi_c^z(q(t)) = 1$ for all $t \ge T^\star$.

Proof: See the Appendix. ∎

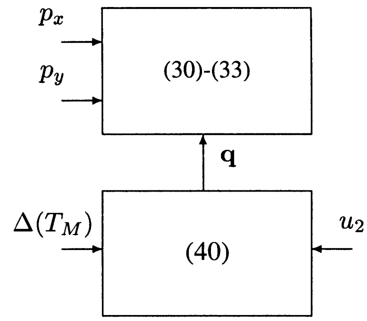

Fig. 3. Overall system dynamics for $t \ge T^\star$. The external signals p_x, p_y and $\Delta(T_M)$ are bounded with p_x and p_y asymptotically vanishing.

With the previous results we have been able to show that tuning the control law (37) with K_3 and K_4 sufficiently large and with λ_2 sufficiently small, the condition $\phi_c^z(q) = 1$ is fulfilled in finite time $T^\star$. This, in view of the results established earlier in Section IV, proves that the suggested control law is able to yield one of the two main design goals, i.e.,

$$\lim_{t \to \infty} |z(t) - z^{\mathrm{ref}}(t)| = 0.$$

It remains to show how to fulfill the other goal, which is ultimate boundedness by arbitrarily small bounds of all other position and attitude variables. Note that in the interval $[0, T^\star]$ the lateral and longitudinal dynamics (30)–(33) behave as chains of integrators driven by bounded signals, therefore do not posses *finite escape times*. This, indeed, allows us to restrict the analysis to the system sketched in Fig. 3 on the time interval $t \ge T^\star$. In Fig. 3, the signals p_x and p_y are defined as

$$p_x = n_x(\mathbf{q})y_{\mathbf{z}}(\mathbf{z}, w) \qquad p_y = n_y(\mathbf{q})y_{\mathbf{z}}(\mathbf{z}, w)$$

and, according to the results established in proposition 5.1, $p_x(t)$ and $p_y(t)$ asymptotically decay to zero.

C. Stabilization of the Lateral and Longitudinal Dynamics

The goal is now the design of u_2 in order to stabilize the interconnected system in Fig. 3, and to provide adequate attenuation of the external disturbances p_x, p_y, $\Delta(T_M)$. It should be noted that, as opposite to p_x and p_y which are vanishing, $\Delta(T_M)$ constitutes a nonvanishing perturbation on the attitude dynamics, as it depends on the main thrust T_M, which in steady state is different from zero. For this reason, in general we cannot expect to reject asymptotically the influence of $\Delta(T_M)$ and achieve convergence of the attitude dynamics to $(q, \omega^b) = (0, 0)$.[4] However, we are able to show that the effect of Δ can be rendered *arbitrarily small* by a proper choice of the design parameters. The

[3]Note that, by definition, $L(T_M) = I + A_\Delta(T_M)A_0^{-1}(T_M)$. Thus, the following requirement on $L(T_M) + L^T(T_M)$ is essentially a restriction on the relative variation of $A(T_M)$ with respect to its nominal value $A_0(T_M)$. It indeed holds if $\|A_\Delta(T_M)A_0^{-1}(T_M)\| \le m^\star I$ for some $m^\star < 1$, which is not a terribly restrictive assumption.

[4]Note that, although in steady state $\Delta(T_M)$ is a function of w, an internal model similar to the one developed in Section IV cannot be employed to asymptotically reject the effect of $\Delta(T_M)$. As a matter of fact, the entries of $\Delta(T_M)$ are, in general, rational functions of T_M and a linear immersion does not exist in this case.

controller will be designed using q_1 and q_2 as virtual controls for the $x - y$ dynamics, and then propagating the resulting control law through the attitude dynamics. Keeping in mind that we need to accomplish this goal using bounded controls, an added difficulty is given by the presence of an unknown time-varying coefficient $\tilde{d}(t)$ in (30)–(33).

Saturation functions, on which the control law u_2 described below is based, are functions $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined in the following way. For $n = 1$, $\sigma(s)$ is any differentiable function satisfying

$$\begin{aligned} |\sigma'(s)| := \left|\frac{d\sigma(s)}{ds}\right| &\leq 2 \text{ for all } s \\ s\sigma(s) &> 0 \text{ for all } s \neq 0, \sigma(0) = 0. \\ \sigma(s) &= \text{sgn}(s) \text{ for } |s| \geq 1 \\ |s| &< |\sigma(s)| < 1 \text{ for } |s| < 1. \end{aligned}$$

For $n > 1$

$$\sigma : \text{col}(s_1, \ldots, s_n) \mapsto \text{col}(\sigma(s_1), \ldots, \sigma(s_n)).$$

To remove drifts in the lateral and longitudinal position due to a constant bias in Δ, we begin by augmenting the system dynamics with the bank of integrators

$$\dot{\eta}_x = x \quad \dot{\eta}_y = y \quad \dot{\eta}_q = q_3. \tag{41}$$

Set now

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad P_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

define the following new state variables:

$$\begin{aligned} \zeta_0 &:= \begin{pmatrix} \eta_y \\ \eta_x \end{pmatrix} \quad \zeta_1 := \begin{pmatrix} y \\ x \end{pmatrix} + \lambda_0 \sigma\left(\frac{K_0}{\lambda_0}\zeta_0\right) \\ \zeta_2 &:= \begin{pmatrix} y_2 \\ x_2 \\ \eta_q \end{pmatrix} + P_1 \lambda_1 \sigma\left(\frac{K_1}{\lambda_1}\zeta_1\right) \end{aligned}$$

and fix, for the control law u_2, the following “nested saturated” structure

$$u_2 = -P_2 \lambda_2 \sigma\left(\frac{K_2}{\lambda_2}\zeta_2\right) \tag{42}$$

where K_i and λ_i, $i = 0, 1, 2$, represent design parameters. Note that, by the definition of saturation function, this choice of u_2 renders the constraint (38) fulfilled. Finally, let

$$\zeta_3 := q - u_2 \qquad \zeta_4 := \omega^b + K_3 \zeta_3$$

so that the overall control law (37) can be rewritten in the more compact form

$$\tilde{\mathbf{v}} = -K_4 \zeta_4. \tag{43}$$

In the new coordinates $\zeta = (\zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_4)^T$, the system of Fig. 3, augmented with (41) and the control u_2 provided by (43), can be put in the following form:

$$\begin{aligned} \dot{\zeta}_0 &= -\lambda_0 \sigma\left(\frac{K_0}{\lambda_0}\zeta_0\right) + \zeta_1 \\ \dot{\zeta}_1 &= -\lambda_1 \sigma\left(\frac{K_1}{\lambda_1}\zeta_1\right) + P_0 \zeta_2 + K_0 \sigma'\left(\frac{K_0}{\lambda_0}\zeta_0\right)\dot{\zeta}_0 \\ M\dot{\zeta}_2 &= -\tilde{D}(t) P_2 \lambda_2 \sigma\left(\frac{K_2}{\lambda_2}\zeta_2\right) \\ &\quad + M K_1 P_1 \sigma'\left(\frac{K_1}{\lambda_1}\zeta_1\right)\dot{\zeta}_1 + \tilde{D}(t)\zeta_3 + p \\ \dot{\zeta}_3 &= \frac{1}{2}[q_0 I + S(q)](\zeta_4 - K_3 \zeta_3) - \dot{u}_2 \\ J\dot{\zeta}_4 &= -S(\omega^b) J (\zeta_4 - K_3 \zeta_3) - K_4 L(T_M) \zeta_4 \\ &\quad + J K_3 \dot{\zeta}_3 + \Delta \end{aligned} \tag{44}$$

where $P_0 = P_1^T$

$$\tilde{D}(t) = \begin{pmatrix} \tilde{d}(t) & m(\mathbf{q}) q_3(t) & 0 \\ m(\mathbf{q}) q_3(t) & -\tilde{d}(t) & 0 \\ 0 & 0 & M \end{pmatrix}$$

and $p = \begin{pmatrix} p_y & p_x & 0 \end{pmatrix}^T$.

The next proposition is the main result of the paper: it shows how, for the control law (43), a proper tuning of the parameters λ_i, $i = 0, 1, 2$, and K_j, $j = 0, 1, 2, 4$ yields input-to-state stability for system (44) with respect to the exogenous inputs p and Δ, with a linear gain with respect to the input Δ which can be rendered arbitrarily small. This means that, since p asymptotically vanishes and Δ is asymptotically bounded by a fixed quantity, the state of the system is ultimately bounded by a quantity that can be rendered arbitrarily small as well. In looking at the next result, it is important to notice that the choice of the design parameter K_3 is dictated by Proposition 5.1 only, and does not play any role in the stabilization procedure. However, since the value of K_3 influences (*but it is not influenced by*) the other design parameters, we assume it fixed once and for all. Furthermore, we make explicitly use of the bounds which, according to the definitions in (32)–(34) and the assumption (12), exist for the functions $m(\mathbf{q}, t)$, and $\tilde{d}(t)$. In particular, we let $M^U > 0$ and d^L be such that

$$|m(\mathbf{q})| \leq M^U, \qquad 0 < d^L \leq \tilde{d}(t)$$

for all $t \geq 0$. Without loss of generality, Proposition 5.1 allows us to assume $q_0(t) > \epsilon > 0$ for all $t > 0$. With this in mind, we have the following result.

Proposition 5.2: Let K_3 be fixed and let $K_i^\star$ and $\lambda_i^\star$, $i = 0, 1, 2$, be such that the following inequalities are satisfied:[5]

$$\frac{\lambda_1^\star}{K_1^\star} < \lambda_0^\star \quad \frac{\lambda_2^\star}{K_2^\star} < \lambda_1^\star \tag{45}$$

[5]It is not difficult to show that numbers $K_i^\star$'s and $\lambda_i^\star$'s satisfying the given inequalities indeed exist (see [13]).

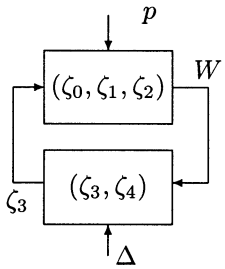

Fig. 4. Closed-loop system as a feedback interconnection.

$$12\frac{K_0^\star}{K_1^\star} < 1 \quad \frac{60}{d^L}\frac{K_1^\star}{K_2^\star} < 1 \tag{46}$$

and

$$M^U \lambda_2^{\star 2} + 4K_1\lambda_1^\star < \frac{\lambda_2^\star}{2} d^L. \tag{47}$$

Then, there exist positive numbers $K_4^\star$, $\epsilon^\star$, R_M, r and γ_Δ such that, taking

$$\lambda_i = \epsilon^i \lambda_i^\star \text{ and } K_i = \epsilon K_i^\star, \qquad i = 0, 1, 2 \tag{48}$$

for all $K_4 \geq K_4^\star$ and $0 < \epsilon \leq \epsilon^\star$, system (44) is ISS with restriction R_M on the input p, no restriction on the input Δ and linear gains $(r, \gamma_\Delta/K_4)$. In particular, if $\|p(\cdot)\|_\infty < R_M$ and $\|\Delta(\cdot)\|_\infty < \infty$, then $\zeta(t)$ exists and is bounded for all $t \geq 0$ and[6]

$$\|\zeta(\cdot)\|_a \leq \max\{r\|p(\cdot)\|_a, \frac{\gamma_\Delta}{K_4}\|\Delta(\cdot)\|_a\}.$$

Proof: System (44) can be seen as the feedback interconnection of two subsystems, as shown in Fig. 4. The upper subsystem is a system with state $(\zeta_0, \zeta_1, \zeta_2)$ and input (p, ζ_3), dynamics described by the first three equations in (44), and output W defined as

$$W = -\dot{u}_2 - K_2 Q \zeta_3 \tag{49}$$

where

$$Q = \frac{1}{M} P_2 \sigma' \left(\frac{K_2}{\lambda_2}\zeta_2\right) \tilde{D}(t).$$

The lower subsystem is a system described by the last two equations in (44) with $-\dot{u}_2$ replaced by $K_2 Q \zeta_3 + W$, that is

$$\begin{aligned} \zeta_3 =& \frac{1}{2}[q_0 I + S(q)](\zeta_4 - K_3\zeta_3) + K_2 Q(t)\zeta_3 + W \\ J\dot{\zeta}_4 =& -S(\omega^b)J(\zeta_4 - K_3\zeta_3) - K_4 L(T_M)\zeta_4 \\ &+ JK_3\dot{\zeta}_3 + \Delta. \end{aligned} \tag{50}$$

It will be shown now that the system in Fig. 4 is a feedback interconnection between ISS systems which satisfy the small gain theorem. First, let us turn our attention to the lower subsystem, for which the following result, proven in [14], holds.

[6]The notation $\|\varphi(\cdot)\|_a$ stands for the *asymptotic norm* of $\varphi(\cdot)$, that is, $\|\varphi(\cdot)\|_a := \limsup_{t\to\infty}\|\varphi(t)\|$.

Lemma 5.1: Let K_3 be fixed and assume that $q_0(t) > \varepsilon > 0$ and $L(T_M(t)) \geq l_1$, for all $t \geq 0$. There exist positive numbers $\bar{\lambda}_2^\star$, $\bar{K}_2^\star(K_3)$, $\bar{K}_4^\star(K_3)$ and r_W, r_Δ such that, for all $\lambda_2 \leq \bar{\lambda}_2^\star$, $K_2 \leq \bar{K}_2^\star(K_3)$ and $K_4 \geq \bar{K}_4^\star(K_3)$, system (50) is ISS, without restriction on the inputs, with linear gains $(r_W, r_\Delta/K_4)$. In particular, if $\|W(\cdot)\|_\infty < \infty$ and $\|\Delta(\cdot)\|_\infty < \infty$, then $(\zeta_3(t), \zeta_4(t))$ exists for all $t \geq 0$ and satisfies

$$\|(\zeta_3(\cdot), \zeta_4(\cdot))\|_a \leq \max\left\{r_W\|W(\cdot)\|_a, \frac{r_\Delta}{K_4}\|\Delta(\cdot)\|_a\right\}.$$

As for the upper subsystem in Fig. 4, the following result, whose proof is again given in [14], holds.

Lemma 5.2: Let λ_i and K_i, $i = 0, 1, 2$, be chosen as in (48) with $\lambda_i^\star$ and $K_i^\star$, satisfying the inequalities (45)–(47). Then, there exist positive $\epsilon^\star$, R_M, r_1 and r_2 such that for all $0 < \epsilon \leq \epsilon^\star$, the output (49) and the state $(\zeta_0, \zeta_1, \zeta_2)$ of system given by the first three in (44) satisfy an asymptotic bound, with nonzero restriction R_M with respect to the input p and restriction $\lambda_2/2$ with respect to the input ζ_3, with linear gains (r_1, r_2) with respect to the state and linear gains $(r_1, r_2 K_2)$ with respect to the output. In particular, if $\|p(\cdot)\|_\infty < R_M$ and $\|\zeta_3(\cdot)\|_\infty < \lambda_2/2$, then $(\zeta_1(t), \zeta_2(t), \zeta_3(t))$ exists for all $t \geq 0$ and satisfies

$$\|W(\cdot)\|_a \leq \max\{r_1\|p(\cdot)\|_a, r_2 K_2\|\zeta_3(\cdot)\|_a\}.$$

The two lemmas contain all that is needed to study the properties of the interconnection in Fig. 4. According to the small gain theorem for ISS systems with restrictions given in [15], the result of the proposition follows if the restriction $\|\zeta_3(t)\| \leq \lambda_2/2$ is satisfied in finite time and the small gain condition

$$r_W r_2 K_2 < 1$$

holds. Without loss of generality, suppose that the number $\epsilon^\star$ in Lemma 5.2 is such that

$$(\epsilon^\star)^2\lambda_2^\star \leq \bar{\lambda}_2^\star(K_3) \quad \epsilon^\star K_2^\star \leq \bar{K}_2^\star(K_3)$$

where $\bar{\lambda}_2^\star(K_3)$, $\bar{K}_2^\star(K_3)$ are those defined in lemma 5.1, so that any choice of λ_2 and K_2 fulfilling (48) with $\epsilon < \epsilon^\star$ also respects the conditions indicated in Lemma 5.1. Using (48), it is seen that the small gain condition is fulfilled if ϵ is sufficiently small so that

$$\epsilon < \frac{1}{r_w r_2 K_2^\star}.$$

As far as the restriction on $\|\zeta_3(t)\|$ is concerned, observe that

$$\begin{aligned} W = K_2 P_2 \sigma'\left(\frac{K_2}{\lambda_2}\zeta_2\right)\Bigg[&-\frac{1}{M}\tilde{D}(t)P_2\lambda_2\sigma\left(\frac{K_2}{\lambda_2}\zeta_2\right) \\ &+ K_1 P_1 \sigma'\left(\frac{K_1}{\lambda_1}\zeta_1\right)\dot{\zeta}_1 + \frac{1}{M}p\Bigg]. \end{aligned}$$

Let $T_1^\star$ be such that $\|p(t)\| \leq R_M$ for all $t \geq T_1^\star$ (such a $T_1^\star$ always exist because $p(t)$ asymptotically decays to zero). Then, simple computations show that $W(t)$, for all $t \geq T_1^\star$, can be bounded by a term which depends only on λ_i and K_i, $i = 0$, 1, 2 and not on $(\zeta_0, \zeta_1, \zeta_2)$. In particular, if λ_i and K_i, $i = 0$, 1, 2 are chosen as in (48), it is possible to claim the existence of numbers $\Gamma_1 > 0$ and $\Gamma_2 > 0$ such that[7]

$$\|W(t)\| \leq \Gamma_1 \epsilon^3 + \Gamma_2 R_M \epsilon \tag{51}$$

for all $t \geq T_1^\star$. Since $\Delta(t)$ is bounded as well, from Lemma 5.1 we see that for any $\rho > 1$ there exists a time $T_2^\star \geq T_1^\star$ such that

$$\|\zeta_3(t)\| \leq \rho \max\left\{ r_W \|W(\cdot)\|_\infty, \frac{r_\Delta}{K_4}\|\Delta(\cdot)\|_\infty \right\}$$

for all $t \geq T_2^\star$ and, hence

$$\|\zeta_3(t)\| \leq 2\rho r_W \left(\Gamma_1 \epsilon^3 + \Gamma_2 R_M \epsilon\right) + 2\rho \frac{r_\Delta}{K_4}\|\Delta(\cdot)\|_\infty \tag{52}$$

for all $t \geq T_2^\star$. The restriction for $\|\zeta_3(t)\|$ is fulfilled on $[T_2^\star, \infty)$ if

$$2\rho r_W \left(\Gamma_1 \epsilon^3 + \Gamma_2 R_M \epsilon\right) + 2\rho \frac{r_\Delta}{K_4}\|\Delta(\cdot)\|_\infty < \frac{\lambda_2}{2}. \tag{53}$$

Again, keeping in mind (48), (53) is fulfilled if the following three conditions are satisfied:

$$2\rho r_W \Gamma_1 \epsilon^3 < \epsilon^2 \frac{\lambda_2^\star}{6} \quad 2\rho r_W \Gamma_2 R_M \epsilon < \epsilon^2 \frac{\lambda_2^\star}{6}$$
$$2\rho \frac{r_\Delta}{K_4}\|\Delta(\cdot)\|_\infty < \epsilon^2 \frac{\lambda_2^\star}{6}.$$

The first inequality can be fulfilled by a sufficiently small ϵ. Once ϵ has been fixed, the second and the third can be satisfied respectively choosing a sufficiently small value for the restriction R_M and a sufficiently large value of K_4.

■

Proposition 5.2 states that there always exists a choice of the design parameters such that the system (44) is ISS with respect to all exogenous inputs, and the gain associated to the input Δ can be rendered arbitrarily small by increasing K_4. Remarkably, this can be done letting the other controller parameters unchanged. It is worth noting that the method relies on *high-gain feedback* as far as the K_4 is concerned, *low-gain feedback* for K_0, K_1 and K_2, and saturation functions whose amplitude λ_i can be chosen arbitrarily small via the scaling parameter ϵ. Since K_4 can be arbitrarily large and λ_2 can be arbitrarily small, the results of this proposition match with those of Proposition 5.1, which indeed required a large value for K_4 and a small value of λ_2. As a consequence, the vertical error dynamics is globally asymptotically stable, which implies that $\|p(\cdot)\|_a = 0$. Therefore, Proposition 5.2 implies that

$$\|\zeta(\cdot)\|_a \leq \frac{\gamma_\Delta}{K_4}\|\Delta(\cdot)\|_\infty.$$

Recall that $\|\Delta(\cdot)\|_\infty$ is bounded by a fixed quantity. Since the value of K_4 can be increased arbitrarily while the other gains K_i, $i = 0, \ldots, 3$ are kept constant, the above result holds for the system in the original coordinates $\left(\mathbf{e}(t), \dot{\mathbf{e}}(t), q(t), \omega^b(t)\right)$ as well. Therefore, we are able to conclude the section stating our final result.

[7]Keeping in mind the expression of W, the bound (51) can be easily obtained using the definition of saturation function, the ϵ-scaling rule in (48) and observing that the quantity $\|\sigma'(K_1\zeta_1/\lambda_1)\sigma'(K_2\zeta_2/\lambda_2)\dot{\zeta}_1\|$ can be upper bounded by a linear function of ϵ. The latter bound can be computed from the expression of $\dot{\zeta}_1$ and $\dot{\zeta}_0$ in (44) assuming without loss of generality that $|\zeta_i| < \lambda_i/K_i$, $i = 1, 2$, as otherwise $\sigma'(K_i\zeta_i/\lambda_i) = 0$.

TABLE I
NOMINAL PARAMETERS OF THE PLANT

$J_x = 0.142413$	$J_y = 0.271256$	$J_z = 0.271492$
$\ell_M = -0.015$	$y_M = 0$	$h_M = 0.2943$
$\ell_T = 0.8715$	$h_T = 0.1154$	$M = 4.9$
$C_M^Q = 0.004452$	$D_M^Q = 0.6304$	$c_M^Q = 25.23$
$C_T^Q = 0.005066$	$D_T^Q = 0.008488$	$c_T^Q = 25.23$

TABLE II
CONTROLLER PARAMETERS

Vertical dynamics	$k_1 = 0.1$	$k_2 = 45$	$\gamma = 1$
Lat./long. dynamics	$K_0 = 0.09$	$K_1 = 0.081$	$K_2 = 0.75$
Attitude dynamics	$K_3 = 0.8$	$K_4 = 30$	$\varepsilon = 0.1$
Saturation levels	$\lambda_0 = 2000$	$\lambda_1 = 8.1$	$\lambda_2 = 0.295$

Theorem 5.1: Consider the dynamic controller given by (13), (36), (19)–(21), and (41)–(43). Let the design parameters be chosen according to Propositions 5.1 and 5.2. Then, for any initial condition $w(0) \in \mathcal{W}$, $\mathbf{z}(0) \in \mathcal{Z}$, $(x(0), \dot{x}(0), y(0), \dot{y}(0)) \in \mathbb{R}^4$, $(q(0), \omega^b(0)) \in \mathcal{Q} \times \Omega$, with $q_0(0) > 0$, the state trajectory in the coordinates $\left(\mathbf{e}(t), \dot{\mathbf{e}}(t), q(t), \omega^b(t)\right)$ is captured by a neighborhood of the origin, which can be rendered arbitrarily small choosing K_4 sufficiently large, and in addition

$$\lim_{t \to \infty} |z(t) - z^{\text{ref}}(t)| = 0.$$

VI. Simulation Results

We present in this section simulation results concerning a specific model of a small unmanned autonomous helicopter described in [6]. The nominal values of the plant parameters are given in Table I. We assume parametric uncertainties up to 20% of the nominal values, therefore we are in presence of a non vanishing perturbing term $\Delta(T_M)$. The oscillatory deck motion is assumed to be generated by a four-dimensional neutrally stable exosystem, with parameters $\varrho = (1, 1.5)$ and initial conditions $w(0) = (3, 1, 2, 3)$. Following Sections IV and V, the controller is designed on the basis of the simplified model of the actuators given by (6) and (7), while simulations are performed on the fully nonlinear actuator model reported in [2]. It should be noted that the presence of unmodeled actuator couplings and parametric uncertainties has the effect of producing a steady-state manifold for the attitude dynamics different from the constant configuration $R(\mathbf{q}) = I$, since it is readily seen from [2] that a time-varying $R(\mathbf{q}(w(t)))$ is needed to offset the vertical steady-state error (see [10]). On the other hand, the presence of nonlinearities in the map $(T_M, \mathbf{v}) \mapsto f^b$ destroys the immersion condition (16), and thus exact asymptotic tracking of $z^{\text{ref}}(t)$ cannot in principle be achieved for $z(t)$. Nevertheless, thanks to the intrinsic robustness of both stabilization methods based on nonlinear versions of the small-gain theorem for ISS systems and internal model based regulation, we expect to be

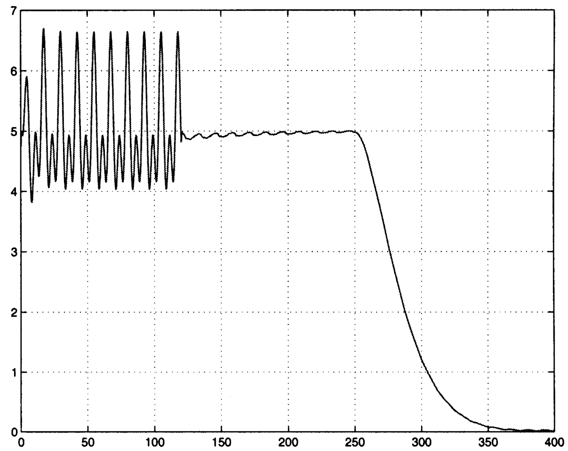

Fig. 5. Tracking error $z(t) - z^{\text{ref}}(t) + H(t)[m]$.

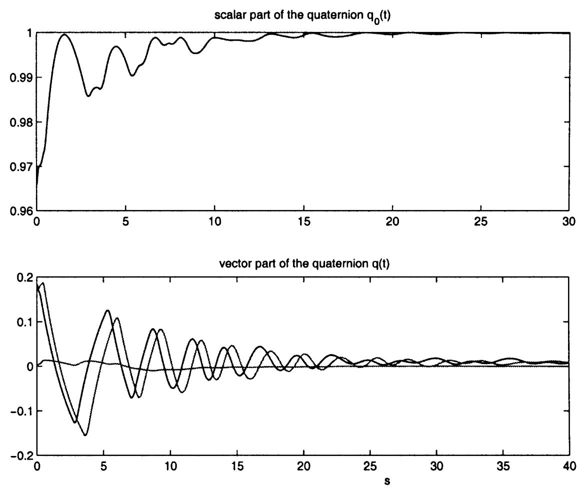

Fig. 6. Quaternions $\mathbf{q}(t)$.

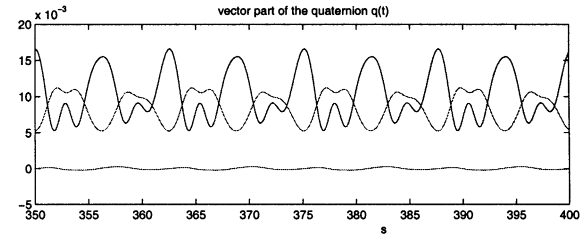

Fig. 7. Steady–state for $q(t)$.

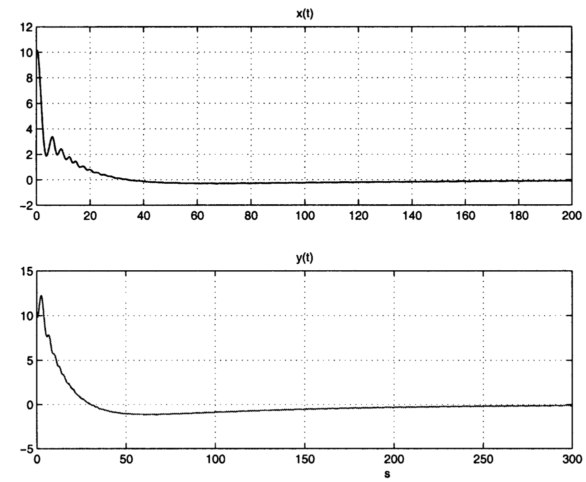

Fig. 8. Longitudinal and lateral displacement $x(t)$, $y(t)[m]$.

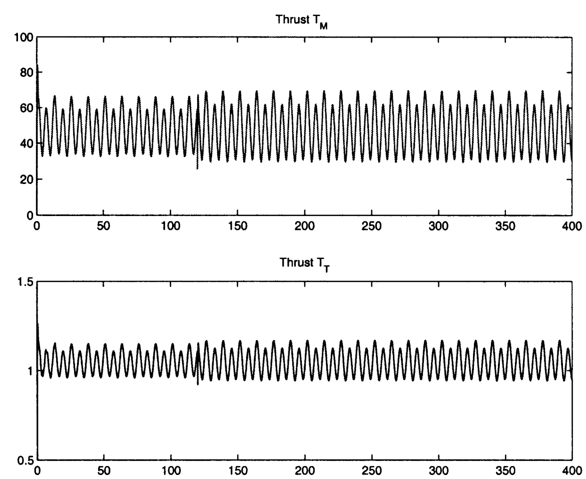

Fig. 9. Main rotor and tail rotor thrusts $T_M(t)$, $T_T(t)[N]$.

able to achieve practical regulation, that is, convergence in finite time to a small neighborhood of the origin for the regulation error $\mathbf{e}(t)$, by a suitable choice of the design parameters.

In all simulations, the control parameters have been selected as in Table II. The vertical bias $H(t)$ has been chosen as

$$H(t) = \begin{cases} 5, & t \in [0,250) \\ 5te^{-0.05(t-250)} & t \geq 250. \end{cases}$$

Initially, the update law for the adaptive internal model has been disconnected, with the natural frequencies of the internal model set at a wrong initial guess $\hat{\varrho} = (1.8, 2)$. Then, the adaptive law has been switched on at time $t = 120$ s. The reported simulation refers to the vehicle initially at rest, with initial attitude and position given by $\mathbf{q}(0) = (0.98, 0.138, 0.138, 0)$ and $(x(0), y(0), z(0)) = (10, 10, 10)$ meters respectively. Fig. 5 shows the vertical error $z(t) - z^{\text{ref}}(t) + H(t)$. The vertical position reaches, in less than 50 s, a sizable steady-state error, due to the initial mismatch of the natural frequencies of the internal model with those of the exosystem. After the adaptation law has been turned on, the vertical error is regulated to $H(t)$, which decreases to zero after time $t = 250$ s. Fig. 6 shows the time history of the attitude parameters. Fig. 7 shows the steady-state response of the attitude parameters $q(t)$: it is readily seen that the vehicle attitude does not converge to $R = I$, as a result of the model uncertainties. As expected, while the attitude dynamics converge rapidly to the steady state (in about 40 s), the lateral and horizontal displacements are brought to zero in a slower time scale (see Fig. 8). The separation of the time scale into a faster and a slower dynamics is a common feature of control laws based on a combination of high-gain and low-amplitude control, as in our case. Finally, Figs. 9 and 10 show the four control variables T_M, T_T and a, b respectively. It is easy to see that the controller succeeds in tracking the unknown reference and in stabilizing the vehicle configuration, despite the large uncertainties on the plant model.

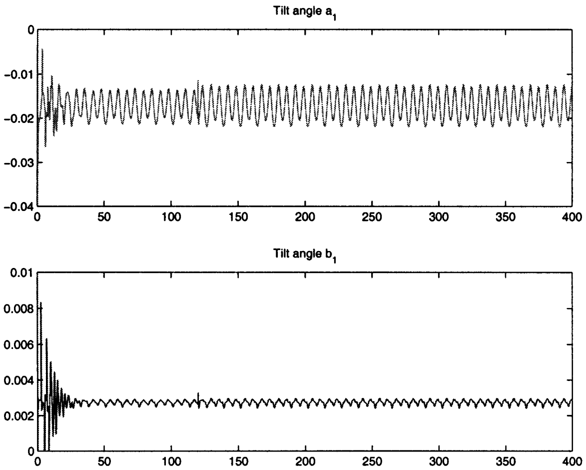

Fig. 10. Tilt angles $a(t)$ and $b(t)$ [rad].

VII. Conclusion and Summary of the Design Method

We have presented an application of nonlinear robust regulation and nonlinear small-gain methods to the challenging problem of designing an autopilot for helicopters landing under uncertain conditions.

In summary, the overall controller is given by a *vertical regulator* yielding the main rotor thrust T_M and an *attitude/lateral/longitudinal stabilizer* computing the input vector $\mathbf{v} = \mathrm{col}(a, b, T_T)$. As far as the vertical regulator is concerned, combining the control laws (13), (19), (20), and (21) yields

$$\begin{aligned}\dot{\xi} =&(F+G\hat{\Psi})\xi - k_2G(\dot{e}_z + k_1e_z) - FGM_0\dot{e}_z\\ \dot{\hat{\Psi}}_2 =& -\gamma\xi_2^T(\dot{e}_z + k_1e_z)\\ T_M =&\frac{gM_0 - \hat{\Psi}\xi + k_2(\dot{e}_z + k_1e_z)}{1-\mathrm{sat}_c(2q_1^2+2q_2^2)}\end{aligned}$$

while the attitude/lateral/longitudinal stabilizer, combining (35), (37) and (41), reads as

$$\begin{aligned}\dot{\eta}_x =& x \quad \dot{\eta}_y = y \quad \dot{\eta}_q = q_3\\ \mathbf{v} =& A_0(T_M)^{-1}\left[-K_4\omega^b - K_4K_3q\right.\\ &\left.+K_4K_3u_2 - B_0(T_M)\right]\end{aligned}$$

where u_2 is the nested saturated control law specified in (42). The overall controller depends on 11 design parameters γ, k_1, k_2, K_i, λ_j with $i = 0, \ldots, 4$, $j = 0, 1, 2$. We have shown that, given arbitrary large compact sets of initial conditions, of uncertain model parameters and of data (frequencies, amplitudes and phases) characterizing the vertical motion of the landing deck, it is possible to tune the design parameters in order to achieve the desired control objective. The overall closed-loop system has dimension $16 + 4N$, where N is the number of sinusoidal signals which approximate the vertical motion of the ship ($N = 2$ in the simulation results). The tuning of the vertical regulator (namely of the parameters γ, k_1 and k_2) has been discussed in Section IV. In particular, while γ and k_1 are arbitrary positive numbers, the value of k_2 must be chosen sufficiently large in order to globally asymptotically stabilize system (23) with $\Phi_c^z(q) = 1$. The tuning of the attitude/lateral/longitudinal stabilizer is indeed more elaborate. In Section V-B a lower bound for K_4 and K_3 and an upper bound for λ_2 have been found (see Proposition 5.1) guaranteeing on one hand that the helicopter never reaches the singular configuration (item a) of the proposition) and, on the other hand, that the condition $\Phi_c^z(q) = 1$ is achieved in finite time (item (b)). The latter achievement guarantees that in finite time the overall system, which is sketched in Fig. 2, behaves as the cascade of the *asymptotically stable* system with state $\mathbf{z}$ driving the attitute/lateral/longitudinal system with state $(y, y_2, x, x_2, q, \omega^b)$ (shown in Fig. 3). Finally the system in Fig. 3 has been shown to be ISS with respect to the input (p_y, p_x) (which is asymptotically decaying) and with respect to the input Δ (with an asymptotic gain which can be rendered arbitrary small by tuning the parameters K_4, K_i and λ_i, $i = 0, 1, 2$). This indeed is the main result of Proposition 5.2.

Appendix

A. Proof of Proposition 5.1

In order to prove Proposition 5.1, we need the following intermediate result

Lemma A.1: Fix compact sets $\mathcal{Z}, \mathcal{W}$ and let T_M^l, T_M^u be such that (39) holds for all $(\mathbf{z}_0, w_0) \in \mathcal{Z} \times \mathcal{W}$, for all q satisfying $\|q\| \le 1$ and for all $t \ge 0$. Let $\bar{\mathbf{z}}(t)$ denote the integral curve of (25) passing through $(\mathbf{z}_0, w_0) \in \mathcal{Z} \times \mathcal{W}$ at time $t = 0$. Let T_0 be such that $\bar{\mathbf{z}}(t)$ is defined on $[0, T_0]$ for all $(\mathbf{z}_0, w_0) \in \mathcal{Z} \times \mathcal{W}$. Then, for any δ there exist $T^\star \le T_0$ such that, if $\phi_c^z(q(t)) = 1$ for all $t \ge T^\star$, $\bar{\mathbf{z}}(t)$ is defined for all $t \ge 0$ and

$$T_M^l - \delta \le \frac{-M\left(\ddot{r}(w) + g\right) - y_{\mathbf{z}}(\bar{\mathbf{z}}, w)}{1 - \mathrm{sat}_c\left(2q_1^2 + 2q_2^2\right)} \le T_M^u + \delta \qquad (54)$$

for all $(\mathbf{z}_0, w_0) \in \mathcal{Z} \times \mathcal{W}$, for all q satisfying $\|q\| \le 1$ and for all $t \ge 0$.

Proof: Consider the compact set $\mathcal{Z}_\varepsilon = \{z : d(z, \mathcal{Z}) \le \varepsilon\}$, where $d(\mathbf{z}, \mathcal{Z})$ denotes the distance of $\mathbf{z}$ from the set $\mathcal{Z}$. Then, bearing in mind the definitions of T_M^l, T_M^u in (39) and the continuity of the functions involved, one can easily see that for any $\delta > 0$ there is $\varepsilon > 0$ such that

$$T_M^l - \delta \le \frac{-M\left(\ddot{r}(w) + g\right) - y_{\mathbf{z}}(\bar{\mathbf{z}}_f, w)}{1 - \mathrm{sat}_c\left(2q_1^2 + 2q_2^2\right)} \le T_M^u + \delta \qquad (55)$$

for all $(\mathbf{z}_0, w_0) \in \mathcal{Z}_\varepsilon \times \mathcal{W}$, for all q satisfying $\|q\| \le 1$ and for all $t \ge 0$. Thus, to prove the lemma, it suffices to show that there is a time $T^\star \le T_0$ such that, for all $x_0 \in \mathcal{Z}$

$$\bar{\mathbf{z}}(t) \in \mathcal{Z}_\varepsilon \qquad \text{for all } t \in [0, T^*].$$

However, this is a simple consequence of the fact that system (25) is locally Lipschitz and that $\mathcal{Z} \times \mathcal{W}$ is a compact set. ■

This lemma essentially guarantees that for any $\delta > 0$ there exist $T^\star$ such that, if $\phi_c^z(q(t)) = 1$ for all $t \ge T^\star$, then the main thrust $T_m(t)$ satisfies

$$T_M^l - \delta \le T_M(t) \le T_M^u - \delta$$

for all $t \ge 0$, for all $(\mathbf{z}(0), w(0)) \in \mathcal{Z} \times \mathcal{W}$ and all $q(0) \in \mathcal{Q}$. Proceeding now with the proof of Proposition 5.1, note that as (q_0, q) and $(-q_0, q)$ represent the same orientation, without loss of generality we can always assume $q_0(0) > 0$ if $q(0) \in \mathcal{Q}$. Since $\mathcal{Q}$ is contained in the open ball of radius $\sqrt{1-\varepsilon^2}$ around the origin in $\mathbb{R}^3$, it turns out that $q_0(0) > \varepsilon$. Change coordinates as

$$\tilde{\omega} := \omega^b + K_3 q$$

and consider the following Lyapunov function candidate:

$$V(q_0, \tilde{\omega}) = \frac{1-q_0}{q_0-\varepsilon} + \frac{1}{2}\tilde{\omega}^T J \tilde{\omega} \tag{56}$$

defined on the set $(\varepsilon, 1] \times \mathbb{R}^3$. Let $0 < \rho < 1$ be such that

$$|q_0| \geq \rho \Rightarrow \phi_a^z(q) \equiv 1.$$

Without loss of generality, assume $\rho > \varepsilon$, and define

$$\vartheta := \frac{1-\rho}{\rho-\varepsilon}.$$

Pick $0 < l_1 < l_1^\star, l_2 > l_2^\star, \delta > \delta^\star$ and, using lemma 1.1, choose $T^\star > 0$ in such a way that, if $\phi_c^z(q(t)) = 1$ for all $t \geq T^\star$, then

$$\begin{aligned} 2l_1 I \leq & L(T_M(t)) + L^T(T_M(t)) \\ \|L(T_M(t))\| \leq & l_2 \quad \|\Delta(T_M(t))\| \leq \delta \end{aligned}$$

for all $t \geq 0$. Let $K_3^\star$ be a positive number such that

$$V\left(q_0(0), \tilde{\omega}(0)\right) \exp\left(-\frac{1-\varepsilon}{2} K_3^\star T^\star\right) \leq \vartheta \tag{57}$$

for all $q(0) \in \mathcal{Q}$ and $\omega^b(0) \in \Omega$. The existence of $K_3^\star$ satisfying (57) is guaranteed by the fact that $V\left(q_0(0), \tilde{\omega}(0)\right)$ is a polynomial function of K_3, and the initial conditions range on a compact set. Fix once and for all $K_3 \geq K_3^\star$, and choose $\mu > \sqrt{\vartheta}$ in such a way that

$$(q, \omega^b) \in \mathcal{Q} \times \Omega \Rightarrow \tilde{\omega} \in \{\tilde{\omega} \in \mathbb{R}^3 : \tilde{\omega}^T J \tilde{\omega} < 2\mu^2\}.$$

Let $c^2 = \max_{q \in \mathcal{Q}}\{1 - q_0/(q_0 - \varepsilon)\}$ and consider the compact sets $\mathcal{S}_{0,\vartheta}$, $\mathcal{S}_{0,c^2+\mu^2}$ and $\mathcal{S}_{\vartheta,c^2+\mu^2}$ where with $\mathcal{S}_{\ell_1,\ell_2}$, $\ell_2 > \ell_1 \geq 0$, we denote

$$\mathcal{S}_{\ell_1,\ell_2} := \left\{(q_0, \tilde{\omega}) \in (\varepsilon, 1] \times \mathbb{R}^3 : \ell_1 \leq V(q_0, \tilde{\omega}) \leq \ell_2\right\}.$$

It is not difficult to see that

$$(q, \omega^b) \in \mathcal{Q} \times \Omega \quad \Rightarrow \quad (q_0, \tilde{\omega}) \in \mathcal{S}_{0,c^2+\mu^2}.$$

Furthermore, if $(q_0, \tilde{\omega}) \in \mathcal{S}_{0,\vartheta}$

$$\frac{1-q_0}{q_0-\varepsilon} \leq \vartheta \quad \Rightarrow \quad q_0 \geq \rho \quad \Rightarrow \quad \phi_c^z(q) = 1.$$

Let us compute the derivative of $V(q_0, \tilde{\omega})$ along trajectories of (40). The first term of (56) yields the following expression:

$$\frac{\partial V(q_0, \tilde{\omega})}{\partial q_0}\dot{q}_0 = \frac{1-\varepsilon}{2(q_0-\varepsilon)^2}\left[q^T\tilde{\omega} - K_3\|q\|^2\right]$$

while the second reads as

$$\begin{aligned} \frac{\partial V(q_0, \tilde{\omega})}{\partial \tilde{\omega}}\dot{\tilde{\omega}} = & -\tilde{\omega}^T S(\tilde{\omega} - K_3 q) J[\tilde{\omega} - K_3 q] \\ & - K_4\tilde{\omega}^T L(T_M)\tilde{\omega} + K_4 K_3 \tilde{\omega}^T L(T_M) u_2 \\ & + \tilde{\omega}^T \Delta(T_M) \\ & + \frac{K_3}{2}\tilde{\omega}^T J[q_0 I + S(q)][\tilde{\omega} - K_3 q]. \end{aligned}$$

Rearranging terms, we obtain

$$\begin{aligned} \dot{V}(q_0, \tilde{\omega}) = & -\frac{1-\varepsilon}{2(q_0-\varepsilon)^2}K_3\|q\|^2 + \\ & \tilde{\omega}^T\left[\frac{1-\varepsilon}{2(q_0-\varepsilon)^2}I - K_3^2 S(q)J - \frac{K_3^2}{2}q_0 J\right]q \\ & + \tilde{\omega}^T\left[K_3 S(q)J - K_4 L(T_M) \right. \\ & \left. + \frac{K_3}{2}J(q_0 + S(q))\right]\tilde{\omega} \\ & + \tilde{\omega}^T\Delta(T_M) + K_3 K_4 \tilde{\omega}^T L(T_M) u_2. \end{aligned} \tag{58}$$

Let c_1, c_2 be such that $0 < c_1 \leq \|J\| \leq c_2$. Since $\|S(q)\| = \|q\|$ and $\|q\| \leq 1$, the derivative of V along solutions of (40) satisfies

$$\begin{aligned} \dot{V}(q_0, \tilde{\omega}) \leq & -K_3\frac{1-\varepsilon}{2(q_0-\varepsilon)^2}\|q\|^2 \\ & + [2K_3 c_2 - K_4 l_1]\|\tilde{\omega}\|^2 \\ & + \left[\frac{3}{2}K_3^2 c_2 + \frac{1-\varepsilon}{2(q_0-\varepsilon)^2}\right]\|\tilde{\omega}\| \\ & + K_3 K_4 l_2 \|\tilde{\omega}\|\|u_2\| + \delta\|\tilde{\omega}\| \end{aligned}$$

for all $(q_0, \tilde{\omega}) \in (\varepsilon, 1] \times \mathbb{R}^3$. What follows is an extension of the results in [16] and [17]. Consider the compact set $\mathcal{S}_{\vartheta,c^2+\mu^2} \cap \{(q_0, \tilde{\omega}) : \tilde{\omega} = 0\}$, and note that on this set

$$\dot{V}(q_0, \tilde{\omega}) < -K_3\frac{1-\varepsilon}{2}V(q_0, \tilde{\omega}). \tag{59}$$

To see that this is indeed the case, it suffices to notice that, on the set $\mathcal{S}_{\vartheta,c^2+\mu^2} \cap \{(q_0, \tilde{\omega}) : \tilde{\omega} = 0\}$

$$\begin{aligned} V(q_0, \tilde{\omega}) = & \frac{1-q_0}{q_0-\varepsilon} \text{ and} \\ \dot{V}(q_0, \tilde{\omega}) \leq & -K_3\frac{1-\varepsilon}{2(q_0-\varepsilon)^2}\left(1-q_0^2\right) \end{aligned}$$

and, thus, (59) holds true if

$$\frac{1-q_0^2}{q_0-\varepsilon} \geq 1 - q_0 \tag{60}$$

which is always satisfied. We prove now that (59) holds everywhere on $\mathcal{S}_{\vartheta,c^2+\mu^2}$. To this end, observe that, by continuity, inequality (59) continues to hold on an open superset $\mathcal{M}$ of $\mathcal{S}_{\vartheta,c^2+\mu^2} \cap \{(q_0, \tilde{\omega}) : \tilde{\omega} = 0\}$. Note that $\mathcal{S}_{\vartheta,c^2+\mu^2}/\mathcal{M}$ is compact and let

$$\begin{aligned} a_1 = & \min_{\tilde{\omega} \in \mathcal{S}_{\vartheta,c^2+\mu^2}/\mathcal{M}} \|\tilde{\omega}\| \quad a_2 = \max_{\tilde{\omega} \in \mathcal{S}_{\vartheta,c^2+\mu^2}/\mathcal{M}} \|\tilde{\omega}\| \\ a_3 = & \max_{q_0 \in \mathcal{S}_{\vartheta,c^2+\mu^2}} \frac{1-\varepsilon}{2(q_0-\varepsilon)^2}. \end{aligned}$$

Needless to say, $a_1 > 0$. Keeping in mind that $\|u_2\|_\infty \le \lambda_2$, we get

$$\dot V(q_0,\tilde\omega) \le -K_3\frac{1-\varepsilon}{2(q_0-\varepsilon)^2}\|q\|^2 + [2K_3c_2 - K_4l_1]\,\|\tilde\omega\|^2 + \left[\frac{3}{2}K_3^2c_2 + a_3 + K_3K_4l_2\lambda_2 + \delta\right]\|\tilde\omega\| \quad (61)$$

for all $(q_0,\tilde\omega) \in \mathcal{S}_{\vartheta,c^2+\mu^2}/\mathcal{M}$. It is easy to prove that there exists a choice of λ_2 and K_4 for which the inequality

$$[2K_3c_2 - K_4l_1]\|\tilde\omega\|^2 + \left[\frac{3}{2}K_3^2c_2 + a_3 + K_3K_4l_2\lambda_2 + \delta\right]\|\tilde\omega\| \le -K_3\frac{1-\varepsilon}{4}c_2\|\tilde\omega\|^2 \quad (62)$$

holds true for all $(q_0,\tilde\omega) \in \mathcal{S}_{\vartheta,c^2+\mu^2}/\mathcal{M}$. In fact (62) holds on $\mathcal{S}/\mathcal{M}$ if

$$K_4(l_1a_1 - K_3l_2\lambda_2) \ge \left(\frac{3}{2}K_3^2c_2 + a_3 + \delta\right) + K_3\varsigma$$

where $\varsigma := c_2(2+1-\varepsilon/4)a_2$ which can be satisfied choosing

$$\lambda_2 \le \lambda_2^\star(K_3) \text{ and } K_4 \ge K_4^\star(K_3)$$

where $\lambda_2^\star(K_3) = l_1a_1/2K_3l_2$ and

$$K_4^\star(K_3) = \frac{3c_2}{a_1l_1}K_3^2 + \frac{(9-\varepsilon)a_2c_2}{2a_1l_1}K_3 + \frac{2(a_3+\delta)}{a_1l_1}.$$

The previous choices for λ_2 and K_4 ensures that

$$\dot V(q_0,\tilde\omega) < -K_3\frac{1-\varepsilon}{2}\left(\frac{\|q\|^2}{(q_0-\varepsilon)^2} + \frac{c_2}{2}\|\tilde\omega\|^2\right)$$

for all $(q_0,\tilde\omega) \in \mathcal{S}_{\vartheta,c^2+\mu^2}$. Moreover, using (60), it is easy to see that

$$\frac{\|q\|^2}{(q_0-\varepsilon)^2} + \frac{c_2}{2}\|\tilde\omega\|^2 \ge \frac{1-q_0}{(q_0-\varepsilon)} + \frac{1}{2}\tilde\omega^TJ\tilde\omega = V(q_0,\tilde\omega)$$

from which we conclude that (59) holds for all $(q_0,\tilde\omega) \in \mathcal{S}$ and, hence, everywhere on $\mathcal{S}_{\vartheta,c^2+\mu^2}$. This result shows that every trajectory $(q(t),\omega^b(t))$ originated within $\mathcal{Q}\times\Omega$ is such that the corresponding trajectory $(q_0(t),\tilde\omega(t))$ is confined inside the positively invariant set $\mathcal{S}_{0,c^2+\mu^2}$, and this proves claim a) of the lemma. To prove claim b), observe that by definition of K_3, we have $V(q_0(T^\star),\tilde\omega(T^\star)) \le \vartheta$ and that $\mathcal{S}_{0,\vartheta}$ also is positively invariant. Since $\phi_c^z(q) = 1$ on $\mathcal{S}_{0,\vartheta}$, the result follows.

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